Quasi-lisse hypertoric VOAs from $3d$ $\mathcal{N} = 4$, free fields, (and holography)

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Introduction and Motivation

Geometry of $4d$ $\mathcal{N} = 2$ VOAs

Recall the $4d$ $\mathcal{N} = 2$ SCFT/VOA correspondence [Beem, Lemos, Liendo, Rastelli, Peelaers, van Rees]

$$\boxed{\mathcal{V} : \{4d\ \mathcal{N} = 2 \text{ SCFT }\} \to \{\text{Vertex Operator Algebra (VOAs) }\}}$$

Given a VOA V, let X_V be the *associated variety* [Arakawa] constructed from Zhu's c_2-algebra [Zhu]

$$X_V := \mathrm{Specm}(R_V)\ ,\ \ V \mapsto R_V := V/c_2(V)$$

Expected geometry of $4d$ $\mathcal{N} = 2$ VOAs

- $X_{\mathcal{V}(\mathcal{T})} \cong \mathcal{M}_H(\mathcal{T})$ (the Higgs branch) as a Poisson variety ($\therefore$ $\mathcal{V}(\mathcal{T})$ is quasi-lisse) [Beem, Rastelli]
- $\mathcal{V}(\mathcal{T})$ is expected to have *free field realisations* modelled on the low-energy theory on $\mathcal{M}_H(\mathcal{T})$ [Beem, Meneghelli, Rastelli]

It is often difficult to systematically construct these free field realisations, unless one works with sheaves of VOAs [Arakawa,Kuwabara,Malikov],[Kuwabara],[Arakawa,Kuwabara,Möller]

Recall the 3d $\mathcal{N}=4$ boundary VOA construction [Costello, Gaiotto], a generalisation of the TQFT/WZW correspondence [Witten]

$$\boxed{\mathcal{V}_A \times \mathcal{V}_B : \{3d\ \mathcal{N}=4 \text{ SQFTs } \} \to \{\text{VOAs}\}_A \times \{\text{VOAs}\}_B}$$

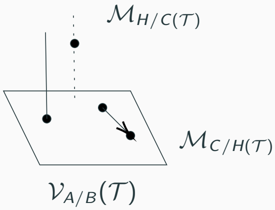
This map depends on boundary degrees of freedom used to cancel gauge anomalies. With appropriate choices (boundary fermions) we conjecturally have that:

Expected geometry of 3d $\mathcal{N}=4$ VOAs

- $X_{\mathcal{V}_{A/B}(\mathcal{T})} \cong \mathcal{M}_{H/C}(\mathcal{T})$ (the Higgs/Coulomb branch) as a Poisson variety ($\therefore$ $\mathcal{V}_{A/B}(\mathcal{T})$ is quasi-lisse) [Beem, AEVF][Coman, Shim, Yamazaki, Zhou]
- As in CS/WZW, bulk lines are expected to form a braided monoidal category of modules $\mathcal{C}_{A/B}(\mathcal{T})$ for the boundary VOA $\mathcal{V}(\mathcal{T})$, and so $\mathrm{Specm}(\mathrm{Ext}_{\mathcal{C}_{A/B}(\mathcal{T})}(\mathbf{1},\mathbf{1})) \sim \mathcal{M}_{C/H}(\mathcal{T})$ [Costello, Creutzig, Gaiotto][Creutzig, Dimofte, Niu]...

Geometry of $3d$ $\mathcal{N} = 4$ VOAs

Thus, in $3d$ there are *two* (Poisson, ...) varieties (schemes, ...) $\mathcal{M}_C(\mathcal{T})$, $\mathcal{M}_H(\mathcal{T})$ controlling VOAs in differently powerful ways. These varieties are expected to be *symplectic dual* pairs. [Braden, Licata, Proudfoot, Webster]...



Remark

- It is necessary for $\mathcal{M}_H(\mathcal{T})$, $\mathcal{M}_C(\mathcal{T})$ to be finite sets of points for $\mathcal{V}_{B/A}(\mathcal{T})$ to be rational.
- Probably also sufficient if $\mathcal{C}_{A/B}(\mathcal{T})$ is rigid.
- New insights on rational non-unitary VOAs [AEVF, Garner, Kim], ... based on [Gang, Yamazaki]...

And of course we have 3d mirror symmetry

$$\{\mathcal{M}_C(\mathcal{T}),\mathcal{M}_H(\mathcal{T})\} \leftrightarrow \{\mathcal{M}_H(\mathcal{T}^\vee),\mathcal{M}_C(\mathcal{T}^\vee)\}$$

Expectations, rephrased

Given a 3d $\mathcal{N}=4$ theory $\mathcal{T}$ with a mirror $\mathcal{T}^\vee$

- One can extract a symplectic dual pair from any single VOA $\mathcal{V}_{\dots}(\mathcal{T}^{\cdots})$ (done in examples [AEVF, Suter])
- Pairs $(\mathcal{V}_A(\mathcal{T}),\mathcal{V}_A(\mathcal{T}^\vee))$ (or equivalently $(\mathcal{V}_B(\mathcal{T}),\mathcal{V}_B(\mathcal{T}^\vee))$) have "mirror features". Associated varieties and identity self-exts are swapped, and so are inner/outer automorphisms.

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This is a remarkable playground. Today, I will focus on some elementary yet (physically, mathematically) interesting examples of *hypertoric* VOAs arising as $\mathcal{V}_A(\mathcal{T})$ for $3d$ Abelian gauge theories $\mathcal{T}$.

Quasi-lisse hypertoric VOAs form 3d $\mathcal{N} = 4$

$3d$ $\mathcal{N}=4$ Abelian Gauge Theories and Hypertoric varieties

$3d$ $\mathcal{N}=4$ Abelian gauge theories are defined by a gauge group $G=U(1)^k$ and a representation T^*R, where we take $R=\mathbb{C}^N$ to be a faithful representation. The Higgs branch is by definition the hypertoric variety [Bielawski, Dancer],[Proudfoot]

$$\mathcal{M}_H := T^*\mathbb{C}^N /\!\!/\!\!/ G \ ,$$

which we view as a holomorphic symplectic quotient

$$\mathcal{M}_H = \mathrm{Spec}[\mathbb{C}[\mu_{\mathbb{C}}^{-1}(0)]^{G_{\mathbb{C}}}] \ .$$

The data defining the theory/representation can be encoded into a (split) SES

$$0 \to \mathfrak{t}_G \xrightarrow{Q^T} \mathfrak{t}_R \xrightarrow{\widetilde{Q}} \mathfrak{t}_F \longrightarrow 0 \ ,$$

where Q contains the weights of the action. Mirror symmetry is Gale duality [Gale],[Proudfoot, Webster]...

$$Q \leftrightarrow \widetilde{Q}$$

Straightforward to compute $\mathcal{M}_C$ and check mirror symmetry [Bullimore, Dimofte, Gaiotto],[Braverman, Finkelberg, Nakajima].

$3d$ $\mathcal{N}=4$ Abelian Gauge Theories and Hypertoric varieties

Here is a fundamental mirror example. Let

$$Q = \begin{pmatrix} 1 & 1 & \cdots 1 \end{pmatrix} \ , \ \widetilde{Q} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & & \end{pmatrix}$$

Then

$$\mathcal{M}_H(\mathcal{T}) \cong \bar{\mathbb{O}}_{\min}(\mathfrak{sl}(N,\mathbb{C})^*) \ , \ \mathcal{M}_C(\mathcal{T}) \cong \mathbb{C}^2/\mathbb{Z}_N.$$

If $\mathbb{C}[T^*\mathbb{C}^N]$ is generated by (X_i, Y^i), then

- $\mathbb{C}[\bar{\mathbb{O}}_{\min}(\mathfrak{sl}(N,\mathbb{C})^*)]$ is generated by rank-1 matrices

  $$M(X,Y) = \{X_i Y^j\}_{ij}$$

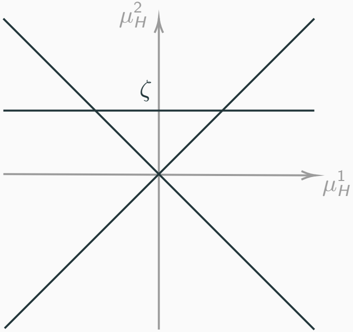
  where $M^2 = 0$ due to $\mu_{\mathbb{C}} = \sum_i X_i Y^i = 0$.
- $\mathbb{C}[\mathbb{C}^2/\mathbb{Z}^N]$ is generated by

  $$W = X_1 X_2 \cdots X_{N-1} \ , \ Z = Y^1 Y^2 \cdots Y^{N-1} \ , \ U = \sum_i X_i Y^i \ .$$

I will use some elementary facts about the geometry of hypertoric varieties. One is the existence of a map

$$\mu_H : \mathcal{M}_H(\mathcal{T}) \to \mathfrak{t}_F^* \cong \mathbb{R}^{N-k}$$

with generic fibre $(\mathbb{R}^2 \times S^1)^{N-k}$ that degenerates along hyperplanes normal to the columns of $\widetilde{Q}$, whose location is prescribed by deformation parameters $\zeta \in \mathfrak{t}_G^*$



Another, related fact is that one may find openly embedded subsets of the form

$$T^*(\mathbb{C}^\times)^{N-k} \hookrightarrow \mathcal{M}_{H,\zeta} \ .$$

by considering the reduction of certain subsets $T^*(\mathbb{C}^\times)^N \hookrightarrow T^*(\mathbb{C})^N$.

Lemma

Let $\epsilon \in \{\pm\}^N$, and $T^*U_\epsilon \hookrightarrow T^*\mathbb{C}^N$, $T^*U_\epsilon \cong T(\mathbb{C}^\times)^N$ with the origin of the base/fibre of the i-th copy of $T^*\mathbb{C}$ removed if $\epsilon_i = +/-$. Then

$$T^*U_\epsilon /\!/ G_\mathbb{C} \cong T^*(\mathbb{C}^\times)^{N-k} \ .$$

Moreover, there is a choice of resolution parameter $\zeta \in (\mathfrak{t}_G^*)^k$ such that

$$T^*U_\epsilon /\!/ G_\mathbb{C} \hookrightarrow \mathcal{M}_{H,\zeta} \ .$$

$3d\ \mathcal{N}=4$ Abelian Gauge Theories and Hypertoric VOAs

Given such a theory $\mathcal{T}$, the boundary VOA $\mathcal{V}_A(\mathcal{T})$ can be defined as a super-chiralisation of the Higgs branch symplectic quotient [Costello,Gaiotto]

- Boundary values of bulk fields $(\beta_i, \gamma^i) = (X_i(t=0), Y^i(t=0))$
- Boundary fermions (ξ^i, χ_i) of same charge
- Gauging is performed by introducing k weight $(0,1)$ bc-ghost systems and taking (relative) BRST cohomology with respect to a rank-k Heisenberg algebra $\hat{\mathfrak{j}}$

$$\mathcal{J}^a = \sum_{i=1}^{N} Q_i^a(\beta_i\gamma^i + \xi^i\chi_i)\ ,\ J = \sum_{a=1}^{k} \mathcal{J}^a c_a\ ,\ d = J_0$$

More formally, this can be expressed in terms of the relative semi-infinite cohomology [Voronov]

$$H^{\frac{\infty}{2}+\bullet}\left(\hat{\mathfrak{j}}, \mathfrak{j}_0, \mathrm{Sb}^N \otimes \mathrm{Ff}^N \otimes (bc)^k\right)$$

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Only Heisenberg fields embedded in the free fermions are responsible for a cancellation of the anomaly

$$\mathcal{J}^a_{\mathrm{Heis}} = \sum_{i=1}^{N} Q^a_i(\beta_i\gamma^i + J_{h_i}) \ , \ J = \sum_{a=1}^{k} \mathcal{J}^a_{\mathrm{Heis}} c_a$$

Rank-k Abelian Heisenberg fields Heis_k, have associated variety

$$X_{\mathrm{Heis}_k} \cong \mathbb{C}^k \ ,$$

which parametrises Poisson deformations. Using this anomaly cancellation, in [Kuwabara], Kuwabara constructed sheaves of $\hbar$-adic vertex algebras on families of Poisson deformations of hypertoric varieties

$$\mathcal{M}^{\mathrm{Heis}}_H \to \mathbb{C}^k \ .$$

In general, the global sections of these sheaves *have not* $\mathcal{M}_H$ as an associated variety (even not $\mathcal{M}^{\mathrm{Heis}}_H$), and they are not quasi-lisse.

$3d\ \mathcal{N} = 4$ Abelian Gauge Theories and Hypertoric VOAs

We can use free fermions instead to build sheaves [Arakawa,Kuwabara,Möller],[Coman et al.][Arakawa,AEVF,Möeller in progress], see Sven's talk in a few hours!.

Here we take a shortcut to characterise the global sections [Beem, AEVF] –by using an analogue of the above Lemma, based a chiral equivalent of the localisation $T^*\mathbb{C}^\times \subset T^*\mathbb{C}$ [Friedan, Martinec, Shenker].

Chiral localisation

For $(\sigma, \sigma) = -(\rho, \rho) = 1$, introduce Heisenberg Then

$$\mathfrak{i}_\pm : \mathrm{Sb} \cong \mathcal{D}_{ch}(\mathbb{C}) \hookrightarrow \mathcal{D}_{ch}(\mathbb{C}^\times)$$
$$\mathfrak{i}_+ : (\beta, \gamma) \mapsto (e^{(\rho-\sigma)}, J_\rho e^{-\rho+\sigma})$$
$$\mathfrak{i}_- : (\beta, \gamma) \mapsto (-J_\rho e^{(-\rho+\sigma)}, e^{\sigma-\rho})$$

where we have inverted either β (X) or γ (Y).

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where we have inverted either β (X) or γ (Y).

Images of $\mathfrak{i}_\pm$ are kernel of *screening operators* $\mathfrak{s}_\pm$:

$$\mathcal{D}_{ch}(\mathbb{C}) \hookrightarrow \mathcal{D}_{ch}(\mathbb{C}^\times) \xrightarrow{\mathfrak{s}_\pm} \mathcal{F}_\pm \ , \ \mathfrak{s}_\pm = \mathrm{Res}_{z=0}(e^\rho \cdot) \ .$$

Given $\mathrm{Sb}^N \times \mathrm{Ff}^N$ and a sign vector $\epsilon \in \{\pm\}^N$, we bosonise everything (fermions with $(\gamma, \gamma) = 1$) to N half-lattices and N lattices

$$\mathfrak{i}_\epsilon : \mathrm{Sb}^N \times \mathrm{Ff}^N \hookrightarrow \bigoplus_{m,n \in \mathbb{Z}^N} e^{m\sigma} e^{-m\rho} \otimes e^{n\omega}$$

and subdivide the lattice directions into directions that pair ($\sim Q_i^a \epsilon_i$) and do not pair ($\sim \widetilde{Q}^i \epsilon_i$) with the BRST operator

$$\mathfrak{i}_\epsilon(\mathcal{J}^a) = \sum_{i=1}^{N} Q_i^a \epsilon_i (J_{\sigma_i} + J_{\gamma_i}) \ .$$

Then since the screening charges commute with the BRST operator, we can compute the BRST cohomology on the bosonised space.

- By a theorem of Voronov [Voronov], it follows that the cohomology is concentrated in degree zero
- The bosonic free fields that survive BRST can be interpreted as $\mathcal{D}_{ch}(U_\epsilon) \cong \mathcal{D}_{ch}((\mathbb{C}^\times)^{N-k})$.

3d $\mathcal{N}=4$ Abelian Gauge Theories and Hypertoric VOAs

More formally, but still schematically, the result is as follows

Quasi-lisse Hypertoric VOAs [Beem, AEVF], [Ballin, Creutzig, Niu , Dimofte]

Let

$$H = \mathbb{Z}^N/(Q_\epsilon^T \mathbb{Z}^k \oplus \widetilde{Q}_\epsilon^T \mathbb{Z}^{N-k}) \ .$$

$\mathcal{V}_A(\mathcal{T})$ is the joint kernel of N screening coperators acting on $\mathcal{V}_{FFR}(\mathcal{T})$

$$\mathcal{V}_{FFR}(\mathcal{T}) \cong \bigoplus_{h\in H} \bigoplus_{\substack{m_{//},m_\perp,n_\perp \\ [m_{//}]=[n_{//}]=[n_\perp]}} e^{m_{//}\rho_{//}+m_\perp\rho_\perp} \otimes e^{-n_\perp\sigma_\perp} \otimes e^{m_\perp\omega_\perp}$$

Remark:

- Unlike in 4d, due to the symplicity of the BRST reduction families of free field realisations are computed systematically
- Geometric analyses of these free field realisations support the conjecure $X_{\mathcal{V}_A(\mathcal{T})} \cong \mathcal{M}_H(\mathcal{T})$
- Expressions for the screening operators are fully explicit

At least in examples, it is also possible to write down candidates for (weak) generators. This is because the chiral rings of the Higgs branches are under reasonable control.

- In the chiral rings one finds $N - k$ complex flavour moment maps

$$\mu^a_{H,\mathbb{C}} = \sum_{i=1}^{N} q_i^a X_i Y^i$$

- For any co-character A of the mirror gauge group, a monomial W^A (mirror to mirror monopole operator).

It is possible to find chiral analogues of these, supplemented by odd currents $X_i \xi^i$ and $Y^i \chi_i$ and various replacements $X_i \leftrightarrow \chi_i$, $Y^i \leftrightarrow \xi^i$. Systematic constructions?

SQED[2] Example

SQED[2] Example

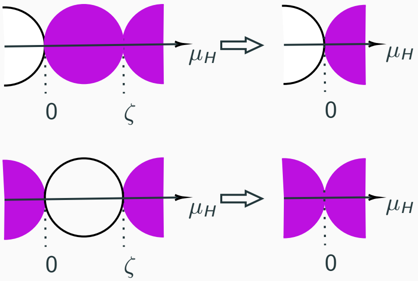
We now exemplify this in the simple example $Q = (1,1)$, *i.e.* SQED[2]. We have

$$\mathcal{M}_H(\text{SQED}[2]) \cong \mathcal{M}_C(\text{SQED}[2]) \cong A_1 \ ,$$

as well as the Springer [Springer] resolution

$$T^*\mathbb{P}^1 \to A_1 \ .$$

The choices $\epsilon = (+-)$ and $\epsilon = (++)$ correspond to two diffent $T^*\mathbb{C}^\times \subset T^*\mathbb{P}^1$, with $(+-)$ not supported on the core (it therefore descends to A_1),



SQED[2] Example

Let us fix $\epsilon = (+-)$

$$\mathcal{J} = \sum_{i=1}^{2} \epsilon_i (J_{\sigma_i} + J_{\gamma_i}) c \ .$$

In this simple example, we can obtain [Creutzig, Linshaw]

$$\mathcal{V}_A(\mathrm{SQED}[2]) \cong V_1(\mathfrak{psl}(2|2))$$

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In fact, naïvely we can take generators cf. [Costello, Creutzig, Gaiotto]

$$\beta_a \gamma^b \ , \ \xi^\alpha \chi_\beta \ , \ \beta_a \xi^\alpha \ , \ \chi_\alpha \gamma^a \ .$$

However:

- $\sum_{i=1}^{2} \epsilon_i (\beta_i \gamma^i - \xi^i \chi_i)$ is not closed
- $\sum_{i=1}^{2} \epsilon_i (\beta_i \gamma^i + \xi^i \chi_i)$ is exact.

SQED[2] Example

From $\beta_i \gamma^j$ we obtain $V_{-1}(\mathfrak{sl}(2)) \subset V_1(\mathfrak{psl}(2|2))$

$$e = e^{(\rho_1+\rho_2)+(-\sigma_1-\sigma_2)} = e^{\delta+\phi}$$

$$h = J_{\sigma_1} + J_{\sigma_2} = -J_\phi$$

$$f = -\bar{J}_{\rho_1}\bar{J}_{\rho_2} e^{-(\rho_1+\rho_2)-(-\sigma_1-\sigma_2)} = \left(-\frac{1}{4}(J_\delta)^2 + T_\psi\right) e^{-\delta-\phi}$$

where T_ψ is a $c = 1$ stress tensor built out of the fermionic trace. Notice

- When $T_\psi = 0$ this reproduces a famous construction by [Adamovic] for $V_k(\mathfrak{sl}(2))$, $k \in \{-\frac{4}{3}, -2, -\frac{1}{2}\}$
- T_ψ comes to the rescue to adjust the level to $V_{-1}(\mathfrak{sl}(2))$
- This fact is key to our free-field realisation of $V_1(\mathfrak{psl}(2|2))$ –it implies that the quotient is implemented on the nose (unlike in other proposals, see [Dei,Gaberdiel, Gopakumar])

SQED[2] Example

Fermions bilinears realise $V_1(\mathfrak{sl}_2)$ as in [Segal, Frenkel-Kac]. The odd generators ($\beta_i \xi^\alpha$ etc.) manifestly carry a $SL(2)_o$ action

$$\Theta^\alpha_{A1} = \psi^\alpha_A e^{\frac{1}{2}(\delta+\phi)}$$

$$\Theta^\alpha_{A2} = \frac{1}{2}\left(\partial\psi^\alpha_A - \psi^\alpha_A \partial\delta + \frac{1}{3}\epsilon_{\beta\gamma}\epsilon^{AB}\psi^\alpha_B \psi^\beta_A \psi^\gamma_C\right) e^{-\frac{1}{2}(\delta+\phi)}$$

Deeper reason: the screening charges can be proven to transform in a representation of this $SL(2)_o$

$$\mathfrak{s}_1 = e^{\frac{1}{2}(\delta+\omega_1+\omega_2)}$$

$$\mathfrak{s}_2 = e^{\frac{1}{2}(\delta+\omega_1-\omega_2)}$$

Manifest outer lemma [Beem, AEVF]

The outer automorphisms of the hypertoric VOA act manifestly on the free fields obtained via a choice ϵ if and only if $T^* U_\epsilon$ is not supported on the core.

Physically, resolutions break Coulomb symmetries.

SQED[2] Example

The choice $\epsilon = (++)$ is different but however also good, for different reasons. With this choice we can "debosonise" the half-lattice

$$\{(\bar{J}_{\rho_2} e^{\phi+\delta}, e^{-\phi-\delta}),\ \mathfrak{s}_2^{(++)}\} \mapsto (\beta, \gamma)\ .$$

to obtain

$$\begin{aligned} e &= \bar{J}_{\rho_2} e^{\phi+\delta} &&= \beta \\ f &= \bar{J}_{\rho_1} e^{-\phi-\delta} &&= (-\beta\gamma + \Psi_i \tilde{\Psi}^i)\gamma \\ h &= -\partial\phi &&= 2\beta\gamma - \Psi_i \tilde{\Psi}^i\ . \end{aligned}$$

for a slightly different choice of fermions. The other screening operator

$$\mathfrak{s}_1^{(++)} = e^{\frac{1}{2}(\delta-\omega_1-\omega_2)}$$

remains to tell us that γ is a coordinate on $\mathbb{P}^1$

$$\gamma(z)\mathfrak{s}_1(w) \sim \mathcal{O}\left(\frac{1}{z-w}\right)\ ,$$

SQED[2] Example

Geometrically, this comes from an open embedding of Recall Ben's talk!

$$T^*\mathbb{C} \cong \mathcal{M}_C(\text{SQED}[1]) \hookrightarrow \mathcal{M}_C(\text{SQED}[2]) \cong \mathcal{M}_H(\text{SQED}[2]) \ .$$

There always is [Recall Ben's talk] a homomorphism from a Coulomb branch to one with less matter. Sometimes, monopole operators can be inverted to turn it into an isomorphism.

Abelain Coulomb perspective

Certain free fields come from embeddings of pure Coulomb branch algebras ($\mathcal{D}_{ch}(\mathbb{C}^\times)$), or SQED[1] ($\mathcal{D}_{ch}(\mathbb{C})$).

This is more broadly related to restriction functors between certain Coulomb branch algebras [Kamnitzer, Webster, Weeks, Yacobi]

SQED[N] example

SQED[N] example

Let now be $N > 3$ and

$$Q = \begin{pmatrix} 1 & 1 & \cdots 1 \end{pmatrix} \ , \ \mathcal{M}_H(\mathcal{T}) \cong \bar{\mathbb{O}}_{\mathrm{min}}(\mathfrak{sl}_N^*) \ .$$

The global sections of [Kuwabara] give in this case

$$\mathcal{V}_A^{\mathrm{Heis}}(\mathrm{SQED}[N]) \cong V_{-1}(\mathfrak{sl}_N) \ .$$

By [Arakawa, Moreau]

$$X_{V_{-1}(\mathfrak{sl}_N)} \cong \bar{\mathbb{S}}_{\mathrm{min}}(\mathfrak{sl}_N) \ ,$$

and

$$\bar{\mathbb{O}}_{\mathrm{min}}(\mathfrak{sl}_N) \subsetneq \bar{\mathbb{S}}_{\mathrm{min}}(\mathfrak{sl}_N) \ .$$

Famously,

$$\bar{\mathbb{S}}_{\mathrm{min}}(\mathfrak{sl}_N) \not\subset \mathcal{N}(\mathfrak{sl}(N)) \ ,$$

and so $\mathcal{V}_A^{\mathrm{Heis}}(\mathrm{SQED}[N])$ is not quasi-lisse. How about $\mathcal{V}_A(\mathcal{T})$?

SQED[N] example

Let us consider $\mathcal{V}_A(\mathrm{SED[N]})$. Using standard results for free fermions/lattices [Segal, Kac][Frenkel], it is easy to see that the even part is

$$V_{-1}(\mathfrak{sl}_N) \oplus V_1(\mathfrak{sl}_N)$$

For $N > 2$, one can check that the BRST computation gives (for $\epsilon = \{+, \cdots, +\}$ say)

$$\mathcal{V}_A(\mathrm{SQED}[N]) \cong \bigoplus_{m\in\mathbb{Z}} \bigoplus_{j=0}^{N-1} L_{-1}(\mathfrak{sl}_N, \lambda(-mN-j)) \oplus L_1(\mathfrak{sl}_N, \omega_j) \ .$$

and using results of [Adamovic,Frajria,Papi,Perše] this gives

SQED[N] Theorem, Part 1 [AEVF,Suter]

$$\mathcal{V}_A(\mathrm{SQED}[N]) \cong V_1(\mathfrak{psl}(N|N))$$

SQED[N] example

Moreover, we can show that $V_1(\mathfrak{psl}(N|N))$ is a super- quasi-lisse extension of $L_{-1}(\mathfrak{sl}_N)$

- For all $N > 1$ we know on general grounds:

$$X_{\mathcal{V}_A(\mathrm{SQED}[N])} \subset \mathfrak{sl}_N^* \times \mathfrak{sl}_N^*$$

- We can show there exists a vector obtained by acting with odd elements on the vacuum

$$v = e^{1,7}_{(-1)} e^{1,8}_{(-1)} |0\rangle$$

generating a submodule U_v containing the maximal submodules of $V^{\pm}(\mathfrak{sl}_N)$, and so

$$X_{\mathcal{V}_A(\mathrm{SQED}[N])} \subset \mathfrak{sl}_N^* \times \{pt.\}$$

- We can show that all 2×2 minors of functions on $\mathfrak{sl}_N^*$ are nilpotent in $V^1(\mathfrak{psl}(N|N))/U_v$, modulo elements in $c_2(V_1(\mathfrak{psl}(N|N)))$!

SQED[N] example

Recall then the Springer resolution [Springer]

$$T^*SL(N,\mathbb{C})/B \cong T^*\mathbb{P}^{N-1} \to \bar{\mathbb{O}}_{\min}(\mathfrak{sl}_N^*) \ .$$

SQED[N] theorem part 2) [AEVF,Suter]

For all $N > 1$ we have

$$X_{V_1(\mathfrak{psl}(N|N))} \cong \bar{\mathbb{O}}_{min}(\mathfrak{sl}_N^*) \ .$$

In particular, $V_1(\mathfrak{psl}(N|N))$ is quasi-lisse, and can be viewed as a chiral quantisation of the algebra of functions on $T^*\mathbb{P}^{N-1}$.

Remarks:

- It follows from [Creutzig,Dimofte,Niu] that self-exts recover $\mathbb{C}^2/\mathbb{Z}_N$
- For the choice of vector $\epsilon \in \{-,+,+,\cdots,+\}$ one can obtain beautiful free field realisations for $\bar{\mathbb{O}}_{\min}(\mathfrak{sl}_N)$ in terms of subsets $(T^*\mathbb{C}^{N-2} \times T^*(\mathbb{C}^\times))/\mathbb{Z}_2$ [Beem, Meneghelli, Rastelli]

A glance at the mirror

The mirror example is supposed to give a fermionic extension of the W algebra $W^{-N+1}(\mathfrak{sl}_N, f_{\mathrm{subreg}})$ [Kuwabara], [Yoshida], [Beem, Ferrari]. For $N = 3$, VOA on $\mathbb{C}^2/\mathbb{Z}^3$ $(\delta,\delta) = -(\phi,\phi) = 3$

$$\begin{aligned} W &\coloneqq e^{\phi+\delta}\,, \\ W^A &\coloneqq \psi^A e^{\frac{2}{3}(\phi+\delta)}\,, \\ W^{AB} &\coloneqq \psi^A\psi^B e^{\frac{1}{3}(\phi+\delta)}\,, \\ W^{ABC} &\coloneqq \psi^A\psi^B\psi^C\,. \end{aligned}$$

$$\begin{aligned} Z &\coloneqq e^{-\phi-\delta}\,, \\ Z_A &\coloneqq \left(\tfrac{1}{9}(\partial\delta + J_\psi)(J_\delta - 2J_\psi)\widetilde{\psi}_A - \tfrac{1}{3}(J_\delta + J_\psi)\widetilde{\psi}'_A - \tfrac{1}{12}\epsilon_{ABC}\epsilon^{DEF}\psi^B\psi^C\widetilde{\psi}_D\widetilde{\psi}_E\widetilde{\psi}_F\right) e^{-\frac{2}{3}(\phi+\delta)}\,, \\ Z_{AB} &\coloneqq \left(-\tfrac{1}{3}\left(\partial\delta + J_\psi\right)\widetilde{\psi}_A\widetilde{\psi}_B + \tfrac{1}{6}\epsilon_{ABC}\epsilon^{DEF}\psi^C\widetilde{\psi}_D\widetilde{\psi}_E\widetilde{\psi}_F\right) e^{-\frac{1}{3}(\phi+\delta)}\,, \\ Z_{ABC} &\coloneqq \widetilde{\psi}_A\widetilde{\psi}_B\widetilde{\psi}_C\,. \end{aligned}$$

$$\begin{aligned} W(z)Z(w) \sim\ & \frac{1}{(z-w)^3} + \frac{J_\phi}{(z-w)^2} + \frac{1}{(z-w)}\left(\frac{1}{2}\partial J_\phi + \frac{1}{3}J_\phi J_\phi + \frac{1}{6}J_\phi J_\psi + \frac{1}{6}J_\psi J_\phi\right) \\ & + \frac{1}{(z-w)}\left(-\frac{1}{6}J_\psi J_\psi + \frac{1}{2}\sum_{A=1}^{3}\left(J^{+,A}J^-_A - J^-_A J^{A,+}\right)\right)\,. \end{aligned}$$

Notice: $SL(3,\mathbb{C})_o$ is manifest.

Minimal tension holography

Associated varieties as a "holographic boundaries"

Recall: whilst a universal affine VOA $V^k(\mathfrak{g})$ can be viewed as a *chiral quantisation* of the algebra of functions on $\mathfrak{g}^*$

$$X_{V^k(\mathfrak{g})} \cong \mathfrak{g}^* \ ,$$

quotients may be chiral quantisations of *subvarieties* of $\mathfrak{g}^*$

$$X_{V_k(\mathfrak{g})} \subset \mathfrak{g}^* \ .$$

When $X_{V_k(\mathfrak{g})} \neq \mathfrak{g}^*$ (and perhaps $V_k(\mathfrak{g})$ is chirally free), this may very loosely be viewed as a *holographic* statement: only a subvariety of $\mathfrak{g}^*$ contributes to the construction of $V_k(\mathfrak{g})$.

Claim

This is less of an April Fools' hoax than it sounds in minimal tension string theory

Minimal tension strings in $AdS_3 \times S^3$

One crucial ingredient of the world-sheet model for string theory on $AdS_3 \times S^3$ in the hybrid formalism [Berkovits, Vafa,Witten] is a WZW model to the supergroup $PSU(1,1|2)$, the isometry group of $AdS_3 \times S^3$.

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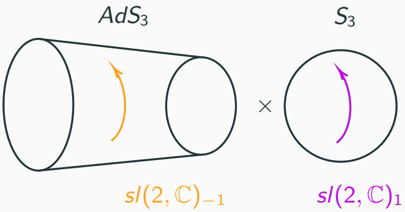

The level of the affine Kac-Moody algebra corresponds to the units of NS-NS flux on S^3. Thus, at minimal flux [Gaberdiel et al.] $k = 1$, the relevant VOA is $V_1(\mathfrak{psu}(1,1|2))$, whose complexification is $V_1(\mathfrak{psl}(2|2))$.

We can represent an element of $SL(2,\mathbb{C})$ as

$$g = e^{\Phi}\begin{pmatrix} e^{-2\Phi} + \gamma\tilde{\gamma} & \gamma \\ \tilde{\gamma} & 1 \end{pmatrix}$$

where γ and $\tilde{\gamma}$ are coordinates on the conformal boundary (complex conjugates in Lorentzian signature). Holomorphic WZW currents can be computed as usual

$$J = g\partial g^{-1} = \begin{pmatrix} \gamma\beta + \partial\Phi & -\gamma^2\beta - \partial\gamma - 2\gamma\partial\Phi \\ \beta & -\gamma\beta - \partial\Phi \end{pmatrix}$$

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Observation

By explicit comparison with our previous FFR, the boundary coordinate γ on $S^2 \cong \mathbb{P}^1$ is identified with a coordinate on $\mathbb{P}^1 \subset \widetilde{X}_{V_1(\mathfrak{psl}(2|2))} \cong T^*\mathbb{P}^1$. The radial coordinate $\partial\Phi = \tilde{\Psi}_i\Psi^i$ is nilpotent!

Minimal tension strings in $AdS_3 \times S^3$

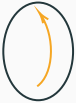
A hallmark of minimal tension string theory on $AdS_3 \times S^3 \times T^4$ is the localisation of the world-sheet path integral onto covering maps from the world-sheet Σ to the boundary S^2 [Eberhardt,Gaberdiel,Gopakumar]

$$\langle \gamma(z) \cdots \rangle_{phys} = \Gamma(z) \langle \cdots \rangle_{phys}$$
$$\Gamma : \Sigma \to S^2 \cong \mathbb{P}^1$$

reproducing the correlators of space-time symmetric product orbifolds e.g. $\mathrm{Sym}^{n\to\infty}(T^4)$ [Lunin, Mathur],[Pakman, Rastelli, Razamat]

- The newest incarnation of the localisation arguments [Dei, Knighton, Naderi] borrowed the above free field realisation. The main ingredient (besides holomorphicity of γ) is a "secret representation" that can be identified with our screening operators $\mathfrak{s}_2^{\epsilon=(++)}$
- Thus the main gadget responsible for the localisation seems to be the associated variety at $k = 1$
- The same can be said for other backgrounds ($X_{\delta_1(2,1,\alpha)} = \bar{\mathbb{O}}_{\min}(\mathfrak{sl}_2^*)$), and possibly in higher dimensions ($T^*\mathbb{P}^3$/twistors!) [Gaberdiel, Gopakumar]

AdS_3 S_3

 $\times$

$sl(2,\mathbb{C})_{-1}$ $sl(2,\mathbb{C})_1$

The End