A pedestrian introduction to the geometry of twisted indices

Andrea Ferrari

Durham University

andrea.ferrari@durham.ac.uk

18.05.2022

Junior Physics and Geometry seminar, Oxford

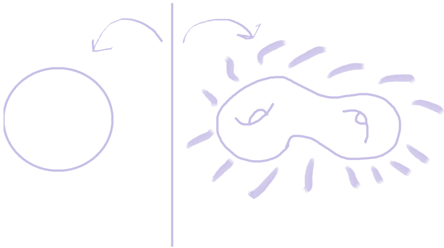
Overview

Introduction and motivation - I

The aim of this talk is to understand twisted indices of 3d $\mathcal{N} \geq 2$ gauge theories from a quantum mechanical viewpoint, and explain rich connections to geometry.

In the first half, we will warm up with the study of QM models with simple targets.

In the second half, we will view 3d $\mathcal{N} = 2$ SUSY gauge theories twisted on $\mathbb{R} \times \Sigma$ surface as a quantum mechanics on $\mathbb{R}$



Introduction and motivation - II

- The Witten index of a quantum mechanics

$$\mathrm{tr}_{\mathcal{H}}(-1)^F e^{-\beta H}$$

receives only contributions from zero-energy states. It is an indicator of e.g. for supersymmetry breaking.

- For a QFT in d dimensions, we can similarly define the Witten index on the torus

$$T^d \cong (S^1)^d \,,$$

which is a graded count of SUSY vacua in flat space.

Introduction and motivation - III

- More generally, twisted indices are partition functions of SUSY theories on manifolds
$$S^1 \times \mathcal{M}^{d-1}\,,$$
additionally graded by flavor symmetries
$$\mathrm{tr}_{\mathcal{H}}(-1)^F e^{-\beta H} y^{J_f}\,.$$
- They can e.g. be used to checks dualities, or to count microstates of blackholes in AdS (via AdS-CFT) [Benini,Hristov,Zaffaroni,...].

Why geometry and why $\mathbb{R}\times\Sigma$?

- More powerful descriptions of SUSY groundstates.
- Connections to beautiful topics in mathematics, such as the Geometric Langlands Program and Symplectic duality [Gaiotto,Kapustin,Witten,..]
- Hilbert spaces on $\mathbb{R}\times\Sigma$ are expected to be isomorphic to conformal blocks of interesting VOAs on Σ [Gaiotto,Costello,Creutzig,Dimofte...].

SUSY algebra and multiplets

A $\mathcal{N} = 2$ quantum mechanics has two supercharges with non-trivial relation

$$\{Q, Q^\dagger\} = H\,.$$

Supersymmetric groundstates $\mathcal{H}$ are configurations annihilated by both Q and $Q^\dagger$. Provided the spectrum of H is gapped,

$$\mathcal{H} \cong H^\bullet_{Q^\dagger}\,.$$

Multiplets

Given a gauge group G, there are three different kinds of multiplets:

- Vector $(\sigma, \lambda, \bar{\lambda}, A, D)$, valued in $Ad(G)$.
- Chiral (ϕ, ψ), valued in a representation V_c of G.
- Fermi (η, F), valued in a representation V_f of G.

From now on, we only consider $G = U(1)$.

Deformations

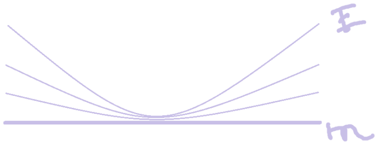
We can and will turn on a real FI parameter $\tau \in \mathfrak{u}(1)^*$ that couples to D via a $\sim \tau(D)$.

In the presence of a flavour isometry G_f, we can also turn on a background vectormultiplet with constant real mass $m \in \mathfrak{t}_f \subset \mathfrak{g}_f$ and background gauge field, which complexify the mass.

This has the effect

$$Q^\dagger \mapsto Q^\dagger_m := e^{-\mu_f \cdot m} Q^\dagger \; e^{\mu_f \cdot m}$$
$$H \mapsto H_m := H - m \cdot J_f \, .$$

Whether the spectrum is gapped may in general depend on m.



Phases

The scalar potential without real masses is

$$U(\sigma,\phi) = |\sigma\phi|^2 + \frac{e^2}{2}\left(\mu_G(\phi) - \tau\right)^2 + \cdots$$

while real masses lead to a shift $\sigma \mapsto \sigma + m$.

We will mainly be concerned with *geometric* phases, where $|\tau| >> 0$ and the gauge group is broken.

Then the SQM has an effective description in terms of a sigma model to

$$\mathcal{M}_H := \{\phi | \mu_G(\phi) = \tau\}/U(1)\,.$$

Often, we can identify

$$Q_m^\dagger = \bar{\partial}_m\,,$$

a deformed Dolbeault operator.

Witten index and localisation - I

The Witten index is defined as

$$\mathcal{I} = \mathrm{tr}_{\mathcal{H}}(-1)^F y^{J_F} \,.$$

Powerful Coulomb-branch localisation formulas have been derived in [Kim,Hori,Yi 2015]. The result for $U(1)$ theories is

$$\mathcal{I} = \frac{1}{2\pi i} \oint_{JK} \frac{\mathrm{d}x}{x} Z_{\mathrm{cl}}(x) Z_{1-\mathrm{loop}}(x, y_i)\,.$$

Here

- $x = e^{-2\pi\beta\sigma + 2\pi i\beta A}$ parametrises the CB $\mathcal{M}_C \cong \mathbb{C}^*$.
- $y_i = e^{-2\pi\beta m_i + 2\pi i\beta\nu_i}$ encodes the background deformations.

Witten index and localisation - II

For N chiral multiplets we have

$$Z_{\text{1-loop, chiral}} = \prod_{i=1}^{N} \left(\frac{(xy_i)^{1/2}}{1 - xy_i} \right) .$$

For N Fermi multiplets, we have

$$Z_{\text{1-loop, Fermi}} = \prod_{i=1}^{N} \left(\frac{1 - xy_i}{(xy_i)^{1/2}} \right) .$$

The classical piece contains contributions from Wilson lines that are x^l for lines of charge l.

The JK residue prescription depends on an auxiliary parameter. If we set $\eta = \tau$, it selects the poles of the chiral muliplets that are positively charged if $\tau > 0$ and negatively if $\tau < 0$.

Free Fermi and chirals

For one chiral with $0 \neq m \in \mathfrak{t}_f$, supersymmetric groundstates are (anti)- hol. functions that are square integrable with respect to $e^{\mp m|\phi|^2}$.

This is the Segal-Bargmann formulation of a complex harmonic oscillator.

$$\mathcal{H}_c = \begin{cases} \{[\phi^n e^{-m|\phi|^2}] \mid n \in \mathbb{N}\} \cong \hat{S}^\bullet(\phi) & m > 0 \\ \{[\bar{\phi}^n e^{m|\phi|^2}\psi] \mid n \in \mathbb{N}\} \cong \hat{S}^\bullet(\bar{\phi}) & m < 0 \end{cases}$$

Here $\hat{S}^\bullet(V) = \det^{1/2}(V) S^\bullet(V)$. This is the correct expansion of the Witten index.

$$\mathcal{I} = \frac{y^{1/2}}{1-y} = y^{1/2}(1 + y + y^2 + y^3 + \cdots)$$

For Fermi multiplets the check is even easier

$$\mathcal{H}_F = \hat{\wedge}^\bullet(\eta)\,,$$

where

$$\hat{\wedge}^\bullet(V) = \det^{-1/2}(V) \wedge^\bullet (V)\,.$$

$\mathbb{C}P^1$ model - I

Consider a $U(1)$ gauge theory with 2 chirals of charge $(+1,+1)$ and a Wilson line of charge l, $l<0$. In the geometric phase $\tau >> 0$ the theory is a sigma model into

$$\mu_G^{-1}(\tau)/U(1) \cong (\mathbb{C}^2 - \{0\})/\mathbb{C}^* \cong \mathbb{CP}^1 \,.$$

The Wilson line generates a bundle on this space, which combined with the charge of the vacuum gives

$$\mathcal{O}(l-1) \to \mathbb{CP}^1 \,.$$

Notice that $l \in \mathbb{Z}$ corresponds to the absence of gauge anomalies (Gauss' law).

By the definition of the index we expect

$$\mathcal{I} = \sum_i H_{\bar\partial}^{\bullet,i}(\mathbb{CP}^1, \mathcal{O}(l-1)) = \chi(\mathbb{CP}^1, \mathcal{O}(l-1)) \,.$$

$\mathbb{C}P^1$ model - II

There are various useful ways to see how this comes about

$$\frac{1}{2\pi i}\oint_{x=y,x=y^{-1}} x^l \frac{1}{1-xy}\frac{1}{1-xy^{-1}} =$$

1) Count homogeneous functions

$$= -\mathrm{Res}_{x=0} x^l (1+xy+(xy)^2+\cdots)(1+xy^{-1}+(xy^{-1})^2+\cdots) = -\sum_{i=0}^{-l-1} y^{l+1+2i}$$

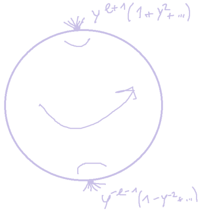
2) Hirzebruch-Riemann-Roch

$$\underset{y\to 1}{=} \mathrm{Res}_{x=1}\left(\frac{x^l}{(1-x)^2}\right) = \int_{\mathbb{CP}^1} \mathrm{Td}(\mathbb{CP}^1)\mathrm{ch}(\mathcal{O}(l-1)) = -l$$

3) Atiyah-Bott-Berline-Vergne fixed-point localisation

$$= \frac{y^{l+1}}{(1-y^2)} + \frac{y^{-l-1}}{(1-y^{-2})} = -\sum_{i=0}^{-l-1} y^{l+1+2i}$$

$$= \sum_{i=1}^{2} \frac{i_{p_i}^* \left(\mathrm{Td}(\mathbb{CP}^1)\mathrm{ch}(\mathcal{O}(l-1))\right)}{e(N_{p_i})}.$$



$\mathbb{C}P^1$ model - Wall-crossing

The first one is really just a trick, but it gives a first hint about the phenomenon of wall-crossing when τ goes from positive to negative.

Notice that $x = 0$ corresponds to $\sigma = -\infty$. The point here is that as we cross from positive to negative, the supersymmetric groundstates start flowing from the Higgs branch to the Coulomb branch and eventually "escape" to infinity.

In this case, all states have escaped, but this is not the general behaviour.

Twisted 3d $\mathcal{N} = 2$ theories

Consider now a 3*d* $\mathcal{N} = 2$ theory. It has a $SU(2)_R$ R-symmetry group. We assume there is an unbroken $U(1)_R$ symmetry group and perform a quasi-topological twist.

Two scalar nilpotent supercharges survive, preserving the SUSY algebra of a $\mathcal{N} = 2$ QM.

We can define the index on $S^1 \times \Sigma$ and consider this as originating from a SUSY QM on $\mathbb{R}$ valued in field configurations on Σ.

In other words, we want to study a sigma model

$$\mathbb{R} \to \text{“Maps”}(\Sigma \to \text{Target})\,.$$

Multiplet decomposition and deformations

A twisted 3d theory comes endowed with a principal $U(1)$ bundle $\mathcal{P}$ on Σ, together with an associated bundle $E = \mathcal{P} \times_\rho V$. The multiplets decompose into 1d multiplets [Bullimore,AF 2018]

3d mutliplet	1d vector	1d chiral	1d Fermi
chiral		$\Omega^{0,0}(E \otimes K^{r/2})$	$\Omega^{0,1}(E \otimes K^{r/2})$
vector	$\Omega^{0,0}(Ad(\mathcal{P}))$	$\Omega^{0,1}(Ad(\mathcal{P}))$	.

- Notice that $D^{1d} \neq D$, but rather $D = D_{1d} - i * F_A$.
- Kinetic terms on the curve are superpotentials from the 1d perspective
- In 3d we obviously have Chern-Simons terms.

Deformations and index

3d theories enjoy deformations as much as 1d theories, and more.

- In addition to m and ν for every flavour symmetry, we can turn on background connections on the curve. Thus, SUSY states must fibre

  $$\mathcal{H} \to \mathrm{Bun}(G_f, \Sigma)\,.$$

  However, we will only mention this for free theories today.
- For each $U(1)$ factor there is a distinguished $U(1)_T$ flavour symmetry, called topological symmetry. For our purposes, it counts the magnetic flux on Σ.

  The corresponding real mass is a 3d FI parameter ξ. It is NOT a 1d FI parameter. This is complexified and exponentiated to a parameter q.

With this in mind, we can define the twisted index as a Witten index for the quantum mechanics

$$\mathcal{I} = \mathrm{tr}_{\mathcal{H}_0}(-1)^F y^{J_f} q^{J_t} = \sum_{\mathfrak{m}\in\mathbb{Z}} q^{\mathfrak{m}} \mathrm{tr}_{\mathcal{H}_{0,\mathfrak{m}}}(-1)^F y^{J_f}\,.$$

Where $\mathcal{H}_{0,\mathfrak{m}}$ is the space of SUSY groundstates with magnetic charge $\mathfrak{m}$.

First localisation scheme

The twisted index in 3d has also been computed using Coulomb-branch localisation scheme, taking $e^2 \to 0$ and $g^2 \to 0$ in [Benini,Zaffaroni 2015 & 2016][Kim,Closset 2016]

$$\mathcal{L} = \frac{1}{e^2}\mathcal{L}_{YM} + \frac{1}{g^2}\mathcal{L}_{\phi} + \mathcal{L}_{\rm CS} + \mathcal{L}_{\rm FI}\,.$$

For a $U(1)$ theory, the classical Coulomb branch is

$$\mathcal{M}_{C,\text{ classical}} \cong \mathbb{C}^*\,.$$

parametrised again by $x = e^{-2\pi\beta\sigma+2\pi i A_0}$. The index reads:

$$\mathcal{I} = \sum_{\mathfrak{m}\in\mathbb{Z}} \frac{1}{2\pi i} q^{\mathfrak{m}} \oint_{JK} \frac{\mathrm{d}x}{x} Z_{\text{classical}}(x) Z_{\text{1-loop},\mathfrak{m}}(x,y_i) H^{g}(x,y_i)\,.$$

where $\mathfrak{m}$ runs over the degree of the gauge bundle.

JK prescription

The contributions are

$$Z_{\text{classical}} = x^{k\mathfrak{m}} x^{l}$$

$$Z_{\text{1-loop},\mathfrak{m}}(x, y_i) = \prod_{i=1}^{N} \left(\frac{\left(x^{Q_i} y_i\right)^{1/2}}{1 - x^{Q_i} y_i} \right)^{Q_i \mathfrak{m} - (r-1)(g-1)}$$

$$H(x, y_i) = k + \sum_{j=1}^{N} Q_j^2 \left(\frac{1}{2} + \frac{x^{Q_j} y_j}{1 - x^{Q_j} y_j} \right)$$

JK depends on an auxiliary parameter $\eta \in \mathfrak{u}(1)^*$. As in 1d, it picks poles of chirals according to sign, but this time assigns charges

$$Q_0 = -k^+_{\text{eff}} \,, \quad Q_\infty = k^-_{\text{eff}} \,.$$

where $k^{\pm}_{\text{eff}}$ are the one-loop ren. k at $\sigma = 0, \infty$.

- This is not really well-defined when $k^{\pm}_{\text{eff}} = 0$, and the prescription is not to take these poles.
- It gives a η-independent answer only after having summed over every $\mathfrak{m}$.

Second localisation scheme and wall-crossing

We would ideally like to have a well-defined prescription for each flux sector $\mathfrak{m}$. Decompose the 3d FI Lagrangian into a 1d FI $\mathcal{L}_\tau$ and another piece $\mathcal{L}'_{\mathrm{FI}}$

$$\mathcal{L}_\tau = -iD_{1\mathrm{d}}\tau$$

[Bullimore,AF,Kim 2019] and consider

$$\mathcal{L} = \frac{1}{t^2}\left(\frac{1}{e^2}\mathcal{L}_{YM} + \mathcal{L}_\tau\right) + \frac{1}{g^2}\mathcal{L}_\phi + \mathcal{L}_{\mathrm{CS}} + \mathcal{L}'_{\mathrm{FI}}\,.$$

In the limit $t^2 \to 0$ we get an identical localisation formula, but with

$$Q_0 = \begin{cases} -k^+_{\mathrm{eff}} & \text{if } k^+_{\mathrm{eff}} \neq 0 \\ \mathfrak{m} - \tau' & \text{else} \end{cases}, \quad Q_\infty = \begin{cases} +k^-_{\mathrm{eff}} & \text{if } k^-_{\mathrm{eff}} \neq 0 \\ \mathfrak{m} - \tau' & \text{else} \end{cases}$$

where $\tau' = \frac{e^2\mathrm{vol}(\Sigma)}{2\pi}\tau$. This clearly depends on τ', and for $\tau' \notin \mathbb{Z}$ is independent of η. In particular,

$$\mathcal{I}(\tau' = \mathfrak{m} + \epsilon) - \mathcal{I}(\tau' = \mathfrak{m} - \epsilon) = q^{\mathfrak{m}}\left[\delta_{0,k^+_{\mathrm{eff}}}\mathrm{Res}_{x=0} - \delta_{k_{\mathrm{eff}-},0}\mathrm{Res}_{x=\infty}\right] I_{\mathfrak{m}}\,.$$

For concreteness, pick $\mathrm{sign}(\eta) = \mathrm{sign}(\mathfrak{m} - \tau')$.

Third localisation scheme

With the previous Lagrangians, we now set $g = t$ and send $t^2 \to 0$ with e^2 finite. The localisation locus becomes

$$* F_A + e^2 \left(\mu_G(\phi) - t^2 \sigma \frac{k_{\text{eff}}(\sigma)}{2\pi} - \tau \right) = 0$$
$$\mathrm{d}_A \sigma = 0\,, \quad \bar{\partial}_A \phi = 0\,, \quad \sigma\phi = 0\,.$$

Schematically, we expect à la HRR

$$\mathcal{I} = \sum_{\mathfrak{m}\in\mathbb{Z}} q^{\mathfrak{m}} \int_{\mathfrak{M}_{\mathfrak{m}}} \hat{\mathrm{A}}(\mathfrak{M}_{\mathfrak{m}}) \mathrm{ch}(\mathcal{F})\,,$$

There are two obvious classes of solutions to the above equation when $\tau' \neq \mathfrak{m}$ [Bullimore,AF,Kim,Xu 2020]:

- Vortex: σ remains finite and so the term with CS level disappears;
- Topological: $t^2\sigma$ remains finite ($|\sigma| \to \infty$) and $\phi = 0$.

Free theories

First, let us understand a free 3d $\mathcal{N} = 2$ chiral multiplet. We had

3d mutliplet	1d vector	1d chiral	1d Fermi
chiral		$\Omega^{0,0}(E \otimes K^{r/2})$	$\Omega^{0,1}(E \otimes K^{r/2})$

where we E is demoted, for free theory, to a background bundle of degree d.

Thus, we have n_c 1d chiral mulitplets and n_F Fermi multiplets with $n_c = h^0(E \otimes K^{1/2})$, $n_F = h^1(E \otimes K^{1/2})$

$$\mathcal{I} = \left(\frac{y^{1/2}}{1 - y} \right)^{n_c - n_F} ,$$

where $n_c - n_F = d + (r - 1)(g - 1)$ depends only on the degree.

Topological saddles - I

We focus on $U(1)_{\frac{1}{2}}$ with one chiral of charge $r = 1$

$$* F_A + e^2 \left(|\phi|^2 - t^2 \sigma \frac{k_{\text{eff}}(\sigma)}{2\pi} - \tau \right) = 0$$
$$\mathrm{d}_A \sigma = 0 \,, \quad \bar{\partial}_A \phi = 0 \,, \quad \sigma \phi = 0$$

where $k_{\text{eff}}^+ = 1$ and $k_{\text{eff}}^- = 0$.

Since $k_{\text{eff}}^+ \neq 0$, consider for $\tau' - \mathfrak{m} < 0$ and $\sigma \to \infty$

$$\tau' - \mathfrak{m} = -\frac{e^2 \mathrm{vol}(\Sigma)}{4\pi^2} \sigma_0 k_{\text{eff}}^+ \,, \quad * F_A = \frac{2\pi \mathfrak{m}}{\mathrm{vol}(\Sigma)} \,.$$

We therefore have

$$\mathfrak{M}_{\mathfrak{m}} = \mathfrak{Pic}^{\mathfrak{m}}(\Sigma) \cong \mathrm{Pic}^{\mathfrak{m}}(\Sigma) \times [\mathrm{pt}/\mathbb{C}^*] \,,$$

In order to integrate over $\mathfrak{M}_{\mathfrak{m}}$,

- We first integrate over $\mathrm{Pic}^{\mathfrak{m}}(\Sigma) := \mathcal{M}_m$
- We then project onto $\mathbb{C}^*$-invariant contributions.

Topological saddles - II

Notice that the vectormultiplet contributions $H^1(\mathrm{Ad}(\mathcal{P}))$ span tangent directions to $\mathcal{M}_\mathfrak{m}$ whereas $H^0(\mathrm{Ad}(\mathcal{P}))$ parametrise the $\mathbb{C}^*$.

- There is an "index bundle" $\mathcal{E}^\bullet$ for the matter contributions, with K-theory class

  $$[\mathcal{E}^\bullet] = [H^0(E \otimes K^{1/2})] - [H^1(E \otimes K^{1/2})]$$

  over the point E. Using the universal line bundle $\mathcal{U} \to \mathcal{M}_\mathfrak{m} \times \Sigma$ and the double fibration

  $$\begin{array}{ccc} & \mathcal{M}_\mathfrak{m} \times \Sigma & \\ {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle p} \\ \mathcal{M}_\mathfrak{m} & & \Sigma \end{array}$$

  we can globalise

  $$\mathcal{E}^\bullet \cong R^\bullet \pi_*(\mathcal{U} \otimes p^* K^{1/2})\,.$$

- A line $\mathcal{L}_{1/2}$ depending on the CS level.

Topological saddles - III

After some manipulations and integration over $\mathcal{M}_{\mathfrak{m}}$, we get

$$\int_{\mathcal{M}_{\mathfrak{m}}} \hat{\mathrm{A}}(\mathcal{M}_{\mathfrak{m}}) \frac{\mathrm{ch}(\mathcal{L}_{1/2})}{\mathrm{ch}(\hat{\wedge}^{\bullet}\mathcal{E})} = x^{1/2\mathfrak{m}} \left(\frac{x^{1/2}}{1-x}\right)^{\mathfrak{m}} \frac{1}{(1-x)^{g}} .$$

This is exactly the integrand computed from localisation! Then we have to project into $\mathbb{C}^*$-invariant expressions. We have $\sigma = \infty$ and so

$$\oint_{x=\infty} \frac{\mathrm{d}x}{x}$$

This is consistend with the assignment of charges from the localisation result!

Upshot: we have a fibration over $\mathcal{M}_{\mathfrak{m}}$. The gaugini span directions tangent to the base, and integration over them yields the Hessian H^{g}. The whole system is gauged.

Vortex saddle - I

For vortex saddles, $\phi \neq 0$ and σ remains finite.

$$*F_A + e^2\left(|\phi|^2 - \tau\right) = 0\,.$$

Integrating over Σ

$$(\tau' - \mathfrak{m}) = ||\phi||^2 > 0\,.$$

Provided this is satisfied, we have an isomorphism

$$\mathfrak{M}_{\mathfrak{m}} \cong \mathrm{Sym}^{\mathfrak{m}}(\Sigma)\,, \quad \mathcal{F} = \mathcal{L}_{1/2}\,,$$

In fact, the zeros of ϕ (center of the vorices) parametrise the moduli space.

$$\sum_{\mathfrak{m}=-\infty}^{\lfloor \tau' \rfloor} q^{\mathfrak{m}} \int_{\mathrm{Sym}^d(\Sigma)} \hat{\mathrm{A}}(\mathrm{Sym}^d(\Sigma))\mathrm{ch}(\mathcal{L}^{1/2}))$$

$$= \sum_{\mathfrak{m}=-\infty}^{\lfloor \tau' \rfloor} \frac{q^{\mathfrak{m}}}{2\pi i} \oint_{x=1} \frac{\mathrm{d}x}{x} \frac{x^{\mathfrak{m}}}{(1-x)^{\mathfrak{m}+g}}\,.$$

Consistent with JK!

Vortex saddles - II

Can we understand this better?

- The massless flutuations around a point of $\mathfrak{M}_{\mathfrak{m}}$ conspire to give

$$0 \to H^0(\mathcal{O}) \to H^0(E \otimes K^{1/2}) \to T\mathrm{Sym}^{\mathfrak{m}}(\Sigma) \to$$
$$\to H^1(\mathcal{O}) \to H^1(E \otimes K^{1/2}) \to 0\,.$$

- There is a map $P : \mathrm{Sym}^{\mathfrak{m}}(\Sigma) \to \mathrm{Pic}^{\mathfrak{m}}(\Sigma)$ so that

$$P^{-1}(L) = \mathbb{P}H^0(E \otimes K^{1/2})\,.$$

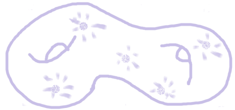
  This is a fibration if $H^1(E \otimes K^{1/2})$=0.

- The "Hessian" part in the integrand comes again from integrating over $\mathrm{Pic}^{\mathfrak{m}}(\Sigma)$, quite literally as in the case of topological vacua.
- The different residue at $x = 1 \leftrightarrow \sigma = 0$ is also intuitive and compatible with the BPS equations: we are setting $\sigma = 0$ and discovering that the gauge symmetry is completly broken (therefore the "$\mathbb{P}$").

Generalisations

Let us consider some generalisations [Bullimore,AF,Kim 2018]:

- We can trade the vortex equation for a stability condition and complex gauge transformations, via the Hitchin-Kobayashi correspondence. What we have seen (producing $\mathrm{Sym}^{\mathfrak{m}}(\Sigma)$) is a simple version of this.
- The stability condition simplifies for $|\tau'| \to \infty$, because the support of the curvature term shrink to points. The outcome is usually a moduli space of quasi-maps to $\mathcal{M}_H$, i.e. pairs (E, X) where of a vector bundle with a section where X fails to hit $\mathcal{M}_H$ at a finite number of points.



- Provided there is a sufficient amount of flavour symmetry, we can then localise to points on $\mathcal{M}_H$, and consider maps to these points. Symmetric products!

3d Mirror Symmetry

3d mirror symmetry is an infrared duality that interchanges

$$y \leftrightarrow q\,.$$

Two of the theories we consider are mirror dual, a free chiral and $U(1)_{1/2}$:

$$U(1)_{1/2} \leftrightarrow 1 \text{ chiral}\,.$$

In order to check this, we take $\tau' \to \infty$. This correponds to the limit of large curve, and so to an IR limit. For the $U(1)_{1/2}$ we have

$$\mathcal{I}[U(1)_{1/2}] = \sum_{\mathfrak{m}=-\infty}^{\infty} \frac{q^{\mathfrak{m}}}{2\pi i} \oint_{x=1} \frac{\mathrm{d}x}{x} \frac{x^{\mathfrak{m}}}{(1-x)^{\mathfrak{m}+g}} = \left(\frac{q}{1-q}\right)^{1-g}\,.$$

$$\mathcal{I}\left[\text{1-chiral}, \text{CS} = -\frac{1}{2}, r=0\right] = \left(\frac{y}{1-y}\right)^{1-g}$$

We can also check agreement of the SUSY groundstates using the geometric picture. Much richer check!

$\mathcal{N}=4$ theories and quasi-maps - I

3d $\mathcal{N}=4$ theories are determined by:

- A gauge group G
- A quaternionic representation $\mathcal{R}_H$ for the matter multiplets.

An important class of theories are quiver gauge theories with $\mathcal{R}_H=\mathcal{R}\oplus\mathcal{R}^*$. The Higgs branch can again be defined as the hyperkähler manifold

$$\mathcal{M}_H := \mu_G^{-1}(\zeta)///G \,.$$

The Coulomb branch of vacua $\mathcal{M}_C$ is more difficult to define properly, but is also is a hyperkḧler manifold. 3d mirror symmetry exchanges

$$\mathcal{M}_C \leftrightarrow \mathcal{M}_H$$

and is mathematically formulated in terms of symplectic duality, a conjectural duality between geometric data associated to mirror pairs of hyperkähler manifolds.

$\mathcal{N}=4$ theories and quasi-maps - II

What can we say about symplectic duality from the twisted index point of view?

- There are two possible kinds of topological twists, A and B.
- Mirror symmetry exchanges the two.
- Let $\mathcal{T}$ be a theory and $\mathfrak{M}^X_{\mathfrak{m}}$ be the space of X-twisted quasi-maps to $\mathcal{M}_H$, $X = A, B$. Schematically, the index is

$$\mathcal{I}[\mathcal{T}, X] = \sum_{\mathfrak{m}} q^{\mathfrak{m}} \int_{\mathfrak{M}^X_{\mathfrak{m}}} \hat{\mathrm{A}}(T\mathfrak{M}^X_{\mathfrak{m}})\,.$$

- Mirror symmetry tells us

$$\mathcal{I}[\mathcal{T}, A](q, y) = \mathcal{I}[\mathcal{T}^\vee, B](y^\vee, q^\vee)\,.$$

Degree-counting and equivariant parameters are exchanged. Highly non-trivial!

The End