Non-invertible symmetries and Higher Representation Theory

Andrea E. V. Ferrari
Based on joint work with T. Bartsch, M. Bullimore, J. Pearson

Symmetry Seminar
18.10.2022

Durham University
andrea.ferrari@durham.ac.uk

Introduction and motivation - I

Recent excitement surrounding non-invertible symmetries:

- Ubiquitous in QFTs

  [Heidenreich, McNamara, et al.] [Kaidi, Ohmori, Zheng] [Bhardwaj, Bottini, Schäfer-Nameki, Tiwari]...

- Physical (including real-world) consequences starting to become manifest.

  [Koide, Nagoya, Yamaguchi[Lin, Okada, Seifnashri, Tachikawa][Choi, Cordova, Hsin, Lam, Shao] ...

Introduction and motivation - I

Recent excitement surrounding non-invertible symmetries:

- Ubiquitous in QFTs

  [Heidenreich, McNamara, et al.] [Kaidi, Ohmori, Zheng] [Bhardwaj, Bottini, Schäfer-Nameki, Tiwari]...

- Physical (including real-world) consequences starting to become manifest.

  [Koide, Nagoya, Yamaguchi[Lin, Okada, Seifnashri, Tachikawa][Choi, Cordova, Hsin, Lam, Shao] ...

In a D-dimensional theory, topological defects are expected to form a $(D-1)$-fusion category. These categories may be quite intricate, which makes it difficult to unleash their full power.

Question

Can one systematically construct theories with non-invertible symmetries whose respective $(D-1)$-fusion categories are completely under control?

Introduction and motivation - II

Various ways to produce non-intrinsically non-invertible symmetries:

- Gauge a discrete symmetry with some 't-Hooft anomaly; [Kaidi, Ohmori, Zheng]...
- Gauge non-normal finite subgroup of a global symmetry; [Arias-Tamargo, Rodriguez-Gomez] [Bhardwaj, Bottini, Schäfer-Nameki, Tiwari] [Antinucci, Galati, Rizi] [Nguyen, Tanizaki, Ünsal] ...
- Gauge a higher-form symmetry along higher co-dimension submanifolds; [Roumpedakis, Seifnashri, Shao]...
- Gauge symmetry along with lower-dimensional TQFT. [Bhardwaj, Schäfer-Nameki, Wu]

Introduction and motivation - II

Various ways to produce non-intrinsically non-invertible symmetries:

- Gauge a discrete symmetry with some 't-Hooft anomaly; [Kaidi, Ohmori, Zheng]...
- Gauge non-normal finite subgroup of a global symmetry; [Arias-Tamargo, Rodriguez-Gomez] [Bhardwaj, Bottini, Schäfer-Nameki, Tiwari] [Antinucci, Galati, Rizi] [Nguyen, Tanizaki, Ünsal] ...
- Gauge a higher-form symmetry along higher co-dimension submanifolds; [Roumpedakis, Seifnashri, Shao]...
- Gauge symmetry along with lower-dimensional TQFT. [Bhardwaj, Schäfer-Nameki, Wu]

Well-known systematic case

[Fröhlich, Fuchs, Runkel, Schweigert][Carqueville, Runkel][Bhardwaj, Tachikawa]...

- Gauge a finite group G in $D = 2$;
- Resulting theory has topological Wilson lines that fuse according to $\mathbf{Rep}(G)$;
- If G is non-abelian obtain non-invertible symmetry category $\mathbf{Rep}(G)$.

General Proposal

D-dimensional theory with finite $(D-1)$-group symmetry G and non-anomalous subgroup $S \subset G \xRightarrow{\text{Gauge } S} (D-1)$-symmetry category related to the higher representation theory of S.

General Proposal

D-dimensional theory with finite $(D-1)$-group symmetry G and non-anomalous subgroup $S \subset G \xRightarrow{\text{Gauge } S} (D-1)$-symmetry category related to the higher representation theory of S.

Today [Bartsch, Bullimore, AEVF, Pearson][1]:

- $D = 3$;
- G finite anomaly-free 2-group;
- $S = G$.

Gauging a 2-group in $D = 3$

Theory $\mathcal{T}$ with non-anomalous, finite 2-group $G \xRightarrow{\text{Gauge } G}$ Theory $\mathcal{T}/G$ with symmetry category $\mathbf{2Rep}(G)$

This applies to a large class of gauge theories.

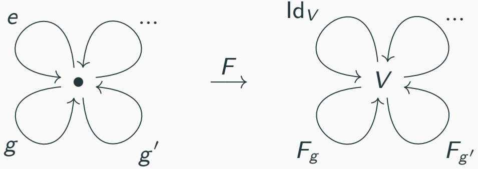
[1] See also [Bhardwaj, Schäfer-Nameki, Wu][Lin,Robbins,Sharpe]

Overview

A Lightning Review of 2-Representations

A Lightning Review of 2-Representations

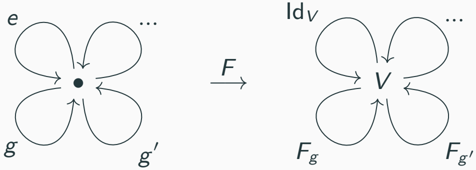
Representations of a group as functors



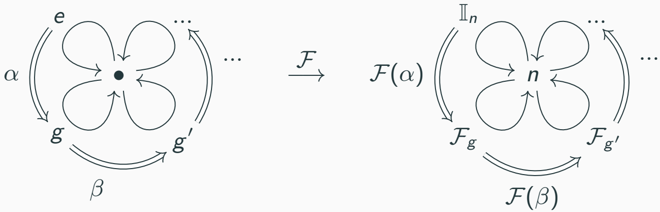
Intertwiners are natural transformations.

A Lightning Review of 2-Representations

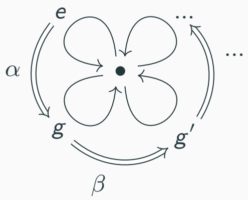
Representations of a group as functors



Intertwiners are natural transformations. Similarly, 2-representations of 2-groups are (pseudo)functors between 2-categories [Baez et al.]



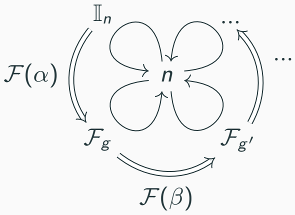
Here



Is a 2-group thought of as an equivalence class of crossed modules

$$0 \to A[1] \to \mathrm{2Hom}\,(e,\ ) \to \mathrm{End}\,(\bullet) \to H \to 0$$

- H is the zero-form part;
- $A[1]$ is the one-form part;
- Can reconstruct action $\rho : H \to \mathrm{Aut}(A[1])$;
- Can reconstruct Postnikov class $\theta \in H^3_\rho\,(H, A[1])$.



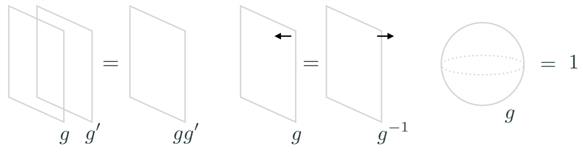
Is a 2-representation $\mathcal{F}$ on the 2-category **2Vect**, up to equivalence [Kapranov-Voedovsky][Osorno]:

- Objects: natural numbers n
- 1-morphisms: $n \times m$ matrices with vector space entries
- 2-morphisms: collections of linear maps between those vector spaces

Gauging Finite Groups

Finite Group Symmetry and 2Vect_G

Consider a theory $\mathcal{T}$ with a finite group symmetry G. Simple objects are surface operators labelled by elements g



A general object is a G-graded set

$$\mathcal{R} = \bigsqcup_{g \in G} \mathcal{R}_g \,,$$

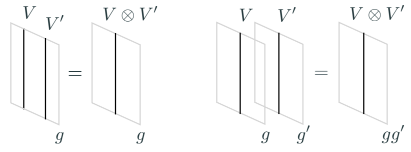
with $\mathcal{R}_g \cong \{1, \ldots, n_g\}$.

$$(\mathcal{R} \oplus \mathcal{R}')_g = \mathcal{R}_g \sqcup \mathcal{R}'_g$$
$$(\mathcal{R} \otimes \mathcal{R}')_g = \bigsqcup_{g = hh'} \mathcal{R}_h \times \mathcal{R}_{h'} \,.$$

1-morphisms are topological lines at junctions between surfaces

$$\mathrm{Hom}_{\mathcal{T}}(g, g') = \begin{cases} \mathbf{Vect} & g = g' \\ 0 & g \neq g' \end{cases}.$$

that is



- 1-morphisms: $n_g \times n'_g$ 2-matrices whose components are vectors spaces
- 2-morphisms: collections of homogeneous linear maps between 2-matrices

This is $\mathbf{2Vect}_G$!

Gauging a Finite Group

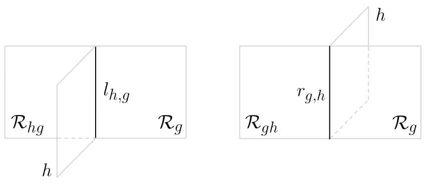
To gauge G, insert algebra object

$$\mathcal{A} = \bigoplus_{h \in G} h \,,$$

Topological surface in $\mathcal{T}/G \Leftrightarrow$ Topological surface in $\mathcal{T}$ + instructions for how $\mathcal{A}$ ends on it inside correlation functions

$$l_{h,g} \in \mathrm{Hom}_{\mathcal{T}}(h \otimes \mathcal{R}_g, \mathcal{R}_{hg})$$

$$r_{g,h} \in \mathrm{Hom}_{\mathcal{T}}(\mathcal{R}_g \otimes h, \mathcal{R}_{gh})$$



Gauging a Finite Group

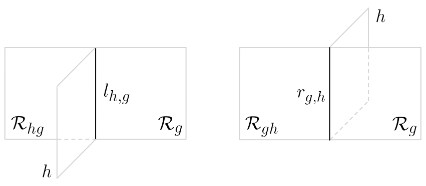
To gauge G, insert algebra object

$$\mathcal{A} = \bigoplus_{h \in G} h \,,$$

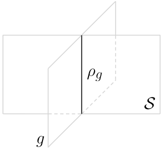
Topological surface in $\mathcal{T}/G$ $\Leftrightarrow$ Topological surface in $\mathcal{T}$ + instructions for how $\mathcal{A}$ ends on it inside correlation functions

$$l_{h,g} \in \mathrm{Hom}_{\mathcal{T}}(h \otimes \mathcal{R}_g, \mathcal{R}_{hg})$$

$$r_{g,h} \in \mathrm{Hom}_{\mathcal{T}}(\mathcal{R}_g \otimes h, \mathcal{R}_{gh})$$



Compatibility conditions best expressed in terms of $\rho_g = (r_{e,g}^{-1} \circ l_{g,e})$



Gauging a Finite Group

Study of compatibility conditions leads to the following classification of topological surfaces in $\mathcal{T}/G$:

1. A set $\mathcal{S} \cong \mathcal{R}_e \cong \{1, \ldots, n\}$;
2. A collection of $n \times n$ 2-matrices $\rho_g = (r_{e,g}^{-1}) \circ l_{g,e} \in \mathrm{Hom}(\mathcal{S}, \mathcal{S})$;
3. A 2-isomorphism $\Psi_e : 1_{\mathcal{S}} \Rightarrow \rho_e$;
4. 2-isomorphisms $\Psi_{g,h} : \rho_{gh} \Rightarrow \rho_g \circ \rho_h$.

With the 2-isomorphisms subject to further conditions

Gauging a Finite Group

Study of compatibility conditions leads to the following classification of topological surfaces in $\mathcal{T}/G$:

1. A set $\mathcal{S} \cong \mathcal{R}_e \cong \{1, \ldots, n\}$;
2. A collection of $n \times n$ 2-matrices $\rho_g = (r_{e,g}^{-1}) \circ l_{g,e} \in \mathrm{Hom}(\mathcal{S}, \mathcal{S})$;
3. A 2-isomorphism $\Psi_e : 1_{\mathcal{S}} \Rightarrow \rho_e$;
4. 2-isomorphisms $\Psi_{g,h} : \rho_{gh} \Rightarrow \rho_g \circ \rho_h$.

With the 2-isomorphisms subject to further conditions

This is precisely an object in **2Rep**(G)! [Elgueta][Osorno]

Gauging a Finite Group

More conveniently, this can be rephrased as

1. A G-set $\mathcal{S}$;
2. A class $c \in H^2(G, U(1)^{\mathcal{S}})$.

What is this more physically?

- If $|\mathcal{S}| = 1$, then $c \in H^2(G, U(1)^{\mathcal{S}})$ determines an SPT phase on the surface.
- If $|\mathcal{S}| > 1$, then these represent condensation defects.

Gauging a Finite Group

More conveniently, this can be rephrased as

1. A G-set $\mathcal{S}$;
2. A class $c \in H^2(G, U(1)^{\mathcal{S}})$.

What is this more physically?

- If $|\mathcal{S}| = 1$, then $c \in H^2(G, U(1)^{\mathcal{S}})$ determines an SPT phase on the surface.
- If $|\mathcal{S}| > 1$, then these represent condensation defects.

To understand this, consider simple objects

$$(\mathcal{S}, c) \;\cong\; \bigoplus_{\alpha} (\mathcal{O}_\alpha, c_\alpha)\,.$$

with

1. a G-orbit $\mathcal{O}$,
2. a class $c \in H^2(G, U(1)^{\mathcal{O}})$.

Gauging a Finite Group

By an orbit-stabilizer correspondence/Shapiro's isomorphism

1. G-orbit $\mathcal{O} \leftrightarrow H \subset G$ subgroup,
2. $c \in H^2(G, U(1)^{\mathcal{O}}) \leftrightarrow c \in H^2(H, U(1))$.

Gauging a Finite Group

By an orbit-stabilizer correspondence/Shapiro's isomorphism

1. G-orbit $\mathcal{O} \leftrightarrow H \subset G$ subgroup,
2. $c \in H^2(G, U(1)^{\mathcal{O}}) \leftrightarrow c \in H^2(H, U(1))$.

Simple objects in 2Rep(G)

Topological surfaces in $\mathcal{T}/G$ where

- Bulk gauge symmetry is broken to $H \subset G$ by partial Dirichlet boundary condition;
- SPT phase $c \in H^2(H, U(1))$.

This is a condensation defect [Roumpedakis Seifnashri Shao].

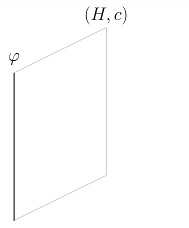
Insight corroborated by fusion of lines on simple surfaces (see later).

- Fusion reproduces Mackey formulas for induction and restriction:

$$(H_\alpha, c_\alpha) \otimes (H_\beta, c_\beta) = \bigoplus_{[g]\in H_\alpha \backslash G / H_\beta} (H_\alpha \cap H_\beta^g \,, \, c_\alpha \otimes c_\beta^g);$$

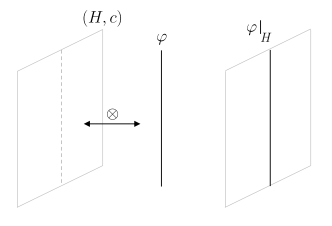
- 1-morphisms reproduce correct behaviour of Wilson lines, e.g.

$$\mathrm{Hom}_{\mathcal{T}/G}(1, (H, c)) \cong \mathrm{Rep}^c(H)$$



- Fusion between $\mathbf{1}_{(H,c)} \in \mathrm{End}((H,c))$ and $\varphi \in \mathrm{End}(1)$:

$$\mathbf{1}_{(H,c)} \otimes \varphi = \varphi|_H \ .$$



Gauging $G = \mathbb{Z}_2$

Elements

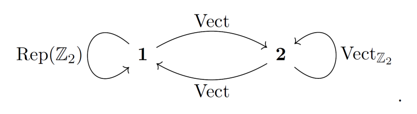
$$\mathbb{Z}_2 \;=\; \{1, x\}\,.$$

Simple 2-representations determined by choice of subgroup $H \subset \mathbb{Z}_2$

	H
1	$\mathbb{Z}_2$
2	$\{1\}$

$\otimes$	**1**	**2**
1	**1**	**2**
2	**2**	$\mathbf{2} \oplus \mathbf{2}$

2 is a condensation defect. 1-morphisms



Gauging Finite 2-Groups

Gauging Finite 2-Group

Can play a similar game starting with a theory $\mathcal{T}$ with a 2-group G

$$G = (H, A[1], \rho, \theta) \ .$$

Gauging Finite 2-Group

Can play a similar game starting with a theory $\mathcal{T}$ with a 2-group G

$$G = (H, A[1], \rho, \theta)\ .$$

In particular, gauging it we obtain topological defects labelled by

- An H-set $\mathcal{S}$ with $|\mathcal{S}| = n$;
- $c \in C^2(H, U(1)^{\mathcal{S}})$;
- $\chi \in (\hat{A})^n$ so that $\partial c = \chi_*(\widetilde{\theta})$, $h \cdot \chi(a) \ = \ \chi(h \cdot a)$;

modulo equivalence relations.

Gauging Finite 2-Group

Can play a similar game starting with a theory $\mathcal{T}$ with a 2-group G

$$G = (H, A[1], \rho, \theta) \ .$$

In particular, gauging it we obtain topological defects labelled by

- An H-set $\mathcal{S}$ with $|\mathcal{S}| = n$;
- $c \in C^2(H, U(1)^{\mathcal{S}})$;
- $\chi \in (\hat{A})^n$ so that $\partial c = \chi_*(\widetilde{\theta})$, $h \cdot \chi(a) \ = \ \chi(h \cdot a)$;

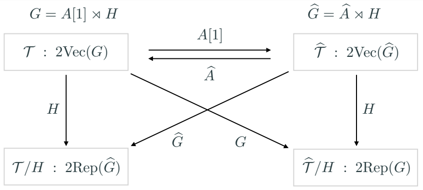
modulo equivalence relations.

These are the objects in **2Rep**(G)! [Elgueta]

This is easiest to describe for G a split 2-group, that is $\theta = 0$

$$G = A[1] \rtimes_\rho H := A[1] \rtimes H$$

Then we can gauge in steps, starting from an ordinary group:



Gauging Finite Split 2-Groups

Symmetry category $\mathcal{T}$ is essentially a higher analogue of a semi-direct-product $\mathbf{Vect}_H$ and $\mathbf{Rep}(\hat{A})$.

Gauging Finite Split 2-Groups

Symmetry category $\mathcal{T}$ is essentially a higher analogue of a semi-direct-product $\mathbf{Vect}_H$ and $\mathbf{Rep}(\hat{A})$.

Simple objects in $\mathcal{T}/G$ are labelled by:

- An H-orbit $\mathcal{O}$ or equivalently a stabilizer subgroup $K \subset H$;
- A class $c \in H^2(K, U(1))$;
- A K-invariant character $\chi \in \hat{A}$.

Gauging Finite Split 2-Groups

Symmetry category $\mathcal{T}$ is essentially a higher analogue of a semi-direct-product $\mathbf{Vect}_H$ and $\mathbf{Rep}(\hat{A})$.

Simple objects in $\mathcal{T}/G$ are labelled by:

- An H-orbit $\mathcal{O}$ or equivalently a stabilizer subgroup $K \subset H$;
- A class $c \in H^2(K, U(1))$;
- A K-invariant character $\chi \in \hat{A}$.

Remarks:

- Fusion, 1-morphisms, composition of 1-morphisms computed as as for an ordinary group (induction-restriction);
- 1-morphisms can only exist between simple objects if and only if the respective characters lie in the same H-orbits in $\hat{A}$;
- Condensation determined by multiplicity of χ.

Example: $(\mathbb{Z}\times\mathbb{Z})[1]\rtimes\mathbb{Z}_2$

Take

$$G = (\mathbb{Z}_2,\, \mathbb{Z}_2\times\mathbb{Z}_2,\, \rho)$$

where $\mathbb{Z}_2 = \{1, x\}$ acts on $\mathbb{Z}_2\times\mathbb{Z}_2$ via

$$x \cdot (a, b) \;:=\; (b, a)\,.$$

- Denote characters of $\mathbb{Z}_2\times\mathbb{Z}_2$ by $\{1,\, \chi_1,\, \chi_2,\, \chi_1\chi_2\}$;
- Label each simple object by pairs $(\mathcal{O}, \chi)$.

Example: $(\mathbb{Z} \times \mathbb{Z})[1] \rtimes \mathbb{Z}_2$

The simple objects are

- the trivial 2-representation $\mathbf{1} = \big(\{1\}, (1)\big)$,
- a 1-dimensional 2-representation $V = \big(\{1\}, (\chi_1\chi_2)\big)$,
- a 2-dimensional 2-representation $D = \big(\{1, 2\}, (\chi_1, \chi_2)\big)$,
- condensation defect $X = \big(\{1, 2\}, (1, 1)\big)$,
- condensation defect $X' = \big(\{1, 2\}, (\chi_1\chi_2, \chi_1\chi_2)\big)$

where condensation is determined by multiplicity of χ.

Example: $(\mathbb{Z} \times \mathbb{Z})[1] \times \mathbb{Z}_2$

Fusion

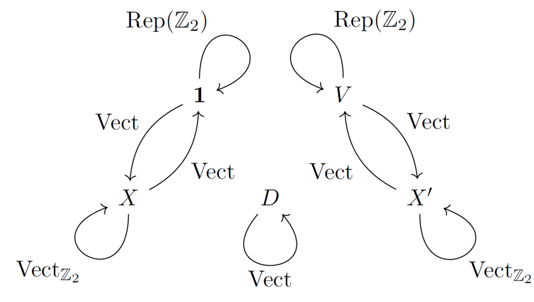
$$V \otimes V = \mathbf{1}$$
$$V \otimes D = D \otimes V = D$$
$$V \otimes X = X'$$
$$D \otimes D = X \otimes (\mathbf{1} \oplus V)$$
$$X \otimes D = D \otimes X = D \oplus D$$
$$X \otimes X = 2X\,.$$

Only invertible simple objects are $\mathbf{1}$ and V. Fusion

$$D \otimes D = X \otimes (\mathbf{1} \oplus V)$$

already implemented at the level of simple objects.

Example: $(\mathbb{Z} \times \mathbb{Z})[1] \times \mathbb{Z}_2$



Gauge theory applications

Gauge theory applications

Simple Lie algebra $\mathfrak{g}$, associated compact, connected, simply connected Lie group $\mathbf{G}$. Automorphism 2-group

$$Z(\mathbf{G})[1] \rtimes \mathrm{Out}(\mathbf{G})$$

where

- $\mathrm{Out}(\mathbf{G})$ outer automorphisms of $\mathbf{G}$
- $Z(\mathbf{G})[1]$ center of $\mathbf{G}$.

and the $\mathrm{Out}(\mathbf{G})$-action outer automorphism action.

Gauge theory applications

Simple Lie algebra $\mathfrak{g}$, associated compact, connected, simply connected Lie group $\mathbf{G}$. Automorphism 2-group

$$Z(\mathbf{G})[1] \rtimes \mathrm{Out}(\mathbf{G})$$

where

- $\mathrm{Out}(\mathbf{G})$ outer automorphisms of $\mathbf{G}$
- $Z(\mathbf{G})[1]$ center of $\mathbf{G}$.

and the $\mathrm{Out}(\mathbf{G})$-action outer automorphism action.

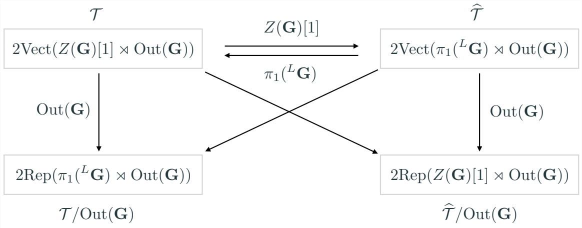
A pure $\mathbf{G}$ gauge symmetry has

- A 0-form charge conjugation symmetry $\mathrm{Out}(\mathbf{G})$.
- A 1-form symmetry $Z(\mathbf{G})$ generated by topological Gukov-Witten defects.

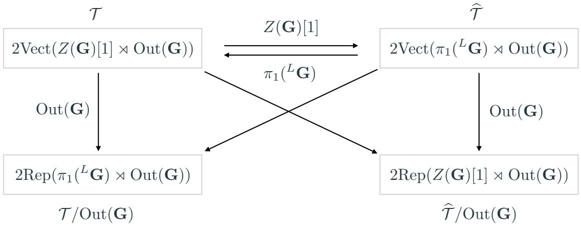
We can gauge this 2-group in steps. The 1-form electric symmetry is traded with a 0-form magnetic symmetry.

Gauge theory applications

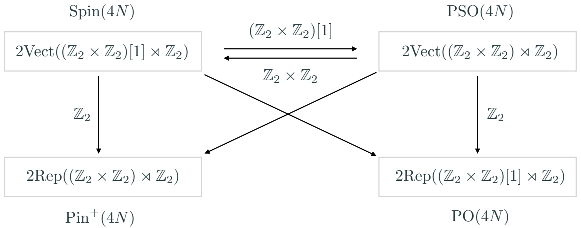
In general:



In general:



$\mathbf{G} = \mathrm{Spin}(4N)$ [Bhardwaj, Bottini, Schäfer-Nameki, Tiwari]:



Conclusions and Outlook

Outlook

Aiming towards [Bartsch, Bullimore, AEVF, Pearson, to appear]

- Theory $\mathcal{T}$ in D dimensions with finite $(D-1)$-group and anomaly $\alpha \in Z^{D+1}(G, U(1))$;
- Symmetry category $\mathbf{Vect}^{\alpha}(G)$;
- Gauge anomaly-free D-subgroup $H \subset G$;
- Resulting category $\mathcal{C}^D(G, \alpha; H, \psi)$ where $\alpha|_H = d\psi$.

Outlook

Aiming towards [Bartsch, Bullimore, AEVF, Pearson, to appear]

- Theory $\mathcal{T}$ in D dimensions with finite $(D-1)$-group and anomaly $\alpha \in Z^{D+1}(G, U(1))$;
- Symmetry category $\mathbf{Vect}^{\alpha}(G)$;
- Gauge anomaly-free D-subgroup $H \subset G$;
- Resulting category $\mathcal{C}^{D}(G, \alpha; H, \psi)$ where $\alpha|_{H} = d\psi$.

Several expected equivalences:

- $\mathcal{C}^{D}(G, \alpha; 1, 1) = (\mathbf{D-1})\mathbf{Vect}^{\alpha}(G)$ for any $(D-1)$-group G.
- $\mathcal{C}^{D}(G, 1; G, 1) = (\mathbf{D-1})\mathbf{Rep}(G)$ for any $(D-1)$-group G.
- ... .

Conclusions

- Higher representation theory provides an efficient framework to investigate non-intrinsically non-invertible symmetries
- This is just the beginning
- The mathematics is being developed in parallel
- This perspective is probably helpful when thinking about actions on non-topological defects (more anon)

The End