Berry connections for 2d $(2,2)$ gauge theories and (generalised) cohomology theories

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Outline

SUSY QFT and Cohomology Theories

For SUSY QFT that flow to a sigma-model with (smooth, compact) target X, it has long been known [Witten,...]

$$\left\{\begin{matrix}\text{Local ops. in } Q\text{-cohomology,} \\ \text{some } Q \text{ s.t. } Q^2 = 0\end{matrix}\right\} \mapsto \left\{\begin{matrix}\text{Some cohomology theory} \\ H^\bullet(X)\end{matrix}\right\}$$

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For instance for a SUSY QM with target X

$$\left\{\begin{array}{l}\{Q, Q^{\dagger}\} = H,\ Q^2 = 0\\ Q\text{-cohomology}\\ \text{SUSY groundstates } \mathcal{H}\end{array}\right\} \mapsto H^{\bullet}_{\mathrm{dR}}(X)$$

Of relevance today

Consider a 2d $(2,2)$ GLSM with a flavor symmetry $T = U(1)^N$ [Hori-Vafa, ...]

$$\left.\begin{cases} Q = Q_A \\ \text{Chiral ring } \mathcal{O}_a\mathcal{O}_b = c_{ab}^{d}\mathcal{O}_d \end{cases}\right\} \mapsto QH_T^{\bullet}(X)$$

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Example

$$H^\bullet(\mathbb{P}^N) \cong \mathbb{C}[\sigma]/\{\sigma^{N+1} = 0\}$$
$$QH^\bullet(\mathbb{P}^N) \cong \mathbb{C}[\sigma, q]/\{\sigma^{N+1} = q\}$$

This can be promoted to a $U(1)^N$-equivariant version

$$QH_T^\bullet(\mathbb{P}^N) \cong \mathbb{C}[\sigma, q]/\left\{\prod_{1=1}^{N+1}(\sigma + m_i) = q\right\}, \quad \sum_{i=1}^{N+1} m_i = 0 \ .$$

SUSY QFT and Cohomology Theories

One can consider an effective theory on the Coulomb branch [Nekrasov-Shatashvili, ...]

$$\left\{ e^{\frac{\partial \widetilde{W}_{eff}(\sigma_1,\cdots,\sigma_k,m_1,\cdots,m_N)}{\partial \sigma_i}} = 1 \right\} \mapsto \{QH^\bullet_T(X) \text{ ring relation}\}$$

where $\sigma_i \in \mathfrak{t}_{G,\mathbb{C}}$, $m_j \in \mathfrak{t}_{\mathbb{C}}$.

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Example:

Take SQED[N+1], a 2d $U(1)$ $(2,2)$ gauge theory with $N+1$ chiral multiplets in the fundamental representation. It flows to a sigma-model to $\mathbb{P}^N$

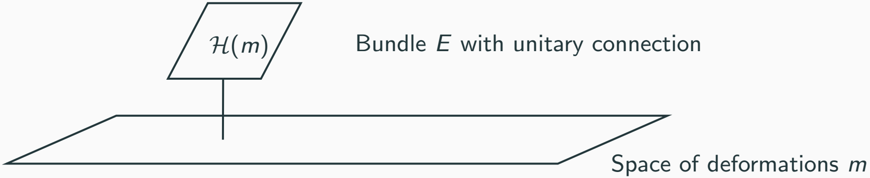
$$\widetilde{W}_{eff}(\sigma, m_1, \cdots, m_{N+1}) = -2\pi i\tau + \sum_{i=1}^{N+1}(\sigma + m_i)(\log(\sigma + m_i) - 1)\ .$$

This flows to a σ-model with target $X = \mathbb{P}^N$

$$1 = e^{-2\pi i\tau}\prod_{i=1}^{N+1}(\sigma + m_i)\ , \quad \sum_{i=1}^{N+1} m_i = 0\ .$$

SUSY QFT and Cohomology Theories

More recently [Bullimore-Zhang, Dedushenko-Nekrasov, ...] there have been attempts to extract *generalised cohomology theories* from *Berry connections*



$$|a(m)\rangle \in \mathcal{H}(m)\ , \quad (A_m)_a^b = \langle a|\,\partial_m\,|b\rangle\ .$$

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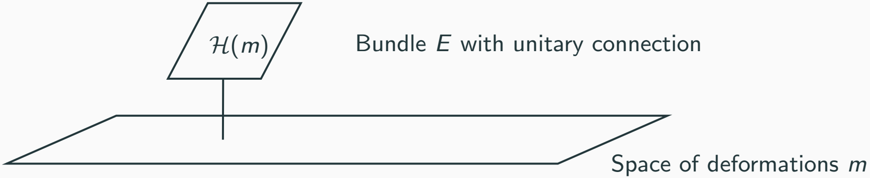

$$|a(m)\rangle \in \mathcal{H}(m) \ , \quad (A_m)_a^b = \langle a| \, \partial_m \, |b\rangle \ .$$

This was worked out for *equivariant elliptic cohomology* starting from a 3d theory on $\mathbb{R} \times \mathcal{E}_\tau$.

SUSY QFT and Cohomology Theories

Today[Ferrari-Zhang]: Given a Kähler manifold X with torus isometries T, one can obtain an analytic object (a generalised periodic monopole) by studying the Berry connection for a 2d $(2,2)$ GLSMs on $S^1_L \times \mathbb{R}$ with target X.

$$\{\text{Kähler manifolds}\} \to \{\text{Generalised periodic monopoles}\}$$

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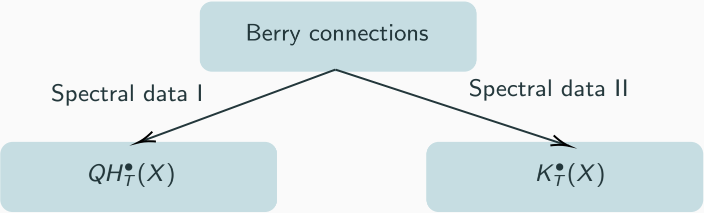
This analytic object encodes data of *different kinds* of (generalised) cohomology theories.



Remark: I will assume $\mathcal{T}$ has isolated massive vacua, and the target X is equivariantly formal (GKM). Some of these assumptions can probably be lifted.

Set-up

Set-up

Consider a theory $\mathcal{T}$ with a rank-one abelian flavor symmetry $T = U(1)$ on a cylinder

We can turn on

- A complex twisted mass $w = w_1 + i w_2$, $(w_1, w_2) \in \mathbb{R}^2$
- a holonomy $t \in \mathbb{R}/\mathbb{Z}L =: S^1_L$ for T.

Space of deformations

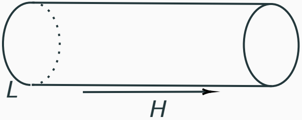
(t, w) are coordinates on

$$M := S^1 \times \mathbb{R}^2 \ ,$$

the space of deformation parameters.

Set-up

Let H be the hamiltonian along $\mathbb{R} \subset \mathbb{R} \times S^1_L$.



Groundstates

Groundstates of H are states $|\alpha\rangle$ s.t. $H|\alpha\rangle = 0$. They form a bundle

$$E \to \mathbb{R}^2 \times S^1 := M \ .$$

The bundle is endowed with a $U(N)$ connection $D = (D_t, D_w, D_{\bar{w}})$ and a Higgs field ϕ s.t. the *Bogomolny equations* hold [Cecotti-Vafa,Cecotti-Gaiotto-Vafa,Tong et al.,...]

$$\star D\phi = F \Rightarrow \begin{cases} [D_{\bar{w}}, D_t - i\phi] = [D_w, D_t + i\phi = 0] \\ 2[D_w, D_{\bar{w}}] = i[D_t, \phi] \end{cases}$$

Set-up

Asymptotics

Asymptotically as $|w| \to \infty$, the following can be proved [Ferrari-Zhang]:

- The bundle E splits as a smooth vector bundle into a sum of line bundles labelled by isolated massive vacua α

$$E = \bigoplus_{\alpha} E_{\alpha}$$

- The combination $A_t + i\phi$ can be computed at each vacuum α in terms of $\widetilde{W}_{\mathit{eff}}$ evaluated at α

$$A_t + i\phi \sim e^{-2i \frac{\partial \widetilde{W}_{\mathit{eff}}}{\partial w}} \Big|_{\alpha}$$

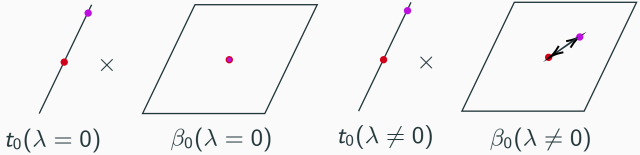
This implies that the monopoles are of so-called Cherkis-Kapustin type; in particular, they tend to copies of $U(1)$ Dirac monopoles as $|w| \to \infty$

Set-up

M is endowed with a $\mathbb{P}^1$ family of *mini-complex structures* [Mochizuki]. More concretely, consider

$$(t_0, \beta_0) := \frac{1}{1+|\lambda|^2}\Big((1-|\lambda|^2)t + 2\mathrm{Im}(\lambda\bar{w}), w + \lambda^2\bar{w} + 2i\lambda t\Big) \ .$$

where λ is a "twistor parameter" on a $\mathbb{C}$ chart of $\mathbb{P}^1$. Intuitively, one can keep in mind the following:

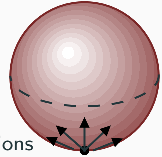


- $\lambda = 0$: $M^0 \cong S^1_t \times \mathbb{C}_w$
- $|\lambda| = 1$: $M^\lambda \cong \mathbb{R}_{t_0} \times \mathbb{C}^*_{\beta_0}$

Set-up

Physically, consider

$$Q_\lambda := \frac{1}{\sqrt{1+\lambda^2}}\left(Q_A + \lambda \bar{Q}_A\right)$$



We have

$$\{Q_\lambda, Q_\lambda^\dagger\} = \{\bar{Q}_\lambda, Q_\lambda^\dagger\} = 2H$$

$$\{Q_\lambda, \bar{Q}_\lambda\} = \frac{1-|\lambda|^2}{1+|\lambda|^2}(2i\partial_2) + 2t_0 \cdot J_T$$

$$Q_\lambda^2 = \frac{2i\lambda\partial_2}{1+|\lambda|^2} - i\beta_0 \cdot J_T$$

Thus, there is a $\mathbb{P}^1$-family of SQMs *adapted to* the mini-complex coordinates.

Set-up

Mochizuki defines a *mini-holomorphic* structure on the bundle

$$E \to M^\lambda$$

In particular, $E_{t_0} := E|_{\{t_0\}\times\mathbb{C}}$ has Dolbeault operator $D_{\bar{\beta}_0}$. Moreover, the Bogomolngy equations imply

$$[D_{t_0} + i\phi, D_{\bar{\beta}_0}] = 0 \ .$$

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Mini-holomorphic ground states [Ferrari-Zhang (see Gaiotto)]

The operators

$$\nabla_{t_0} := D_{t_0} + i\phi \ , \quad \nabla_{\beta_0} := D_{\bar{\beta}_0}$$

descend to Q_λ cohomology

$$\{\nabla_{t_0}, Q_\lambda\} = \{\nabla_{\beta_0}, Q_\lambda\} = 0$$

The supersymmetric ground states viewed as states in Q_λ-cohomology have the structure of a mini-holomoprhic bundle.

Spectral Data I

Spectral Data I, $\lambda = 0$

Recall $M^{\lambda=0} \cong S^1 \times \mathbb{C}$ and define

$$V := H^0(\mathbb{C}_w, E^0)$$

Parallel transport along ∇_{t_0} defines a $\mathbb{C}(w)$-linear automorphism

$$F : \underbrace{H^0(\mathbb{C}_w, E^0)}_{=:V} \to \underbrace{H^0(\mathbb{C}_w, E^L)}_{\cong V}$$

Formally, (F, V) is a 0-difference $\mathbb{C}(w)$-module [Mochizuki]

Mochizuki's correspondence

There is a 1:1 correspondence between (polystable , parabolic , ...) difference modules and periodic Cherkis-Kapustin monopoles.

Spectral Data I, $\lambda = 0$

Moreover one can extract the *spectral curve* [Cherkis-Kapustin]

$$\mathcal{L} := \{(p, w) \in \mathbb{C}^* \times \mathbb{C} \mid \det(p - F(w)) = 0\} \ .$$

This is a Lagrangian subvariety of $\mathbb{C}^* \times \mathbb{C}$ (symplectic form $\Omega = \frac{dp}{p} \wedge dw$)

Spectral curve and quantum cohomology [Ferrari-Zhang]

Suppose that $\mathcal{T}$ flows to a NLSM with GKM target X. Then V is generated by elements in the quantum cohomology $\mathcal{O}_a$

$$|a\rangle := \mathcal{O}_a \, |0\rangle \ .$$

The CK spectral curve for its Berry connection is a "momentum space" representation of

$$\mathrm{Spec}(QH^\bullet_T(X)) \ .$$

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More precisely, (p, w) form a 3d pure gauge theory Coulomb branch algebra that acts on $QH^\bullet_T(X)$, and the spectral curve determines the support of the corresponding sheaf [Ferrari-Zhang, based on Teleman and Bullimore, Dimofte, Gaiotto, Hilburn]

Spectral Data I, $\lambda = 0$

Why and how to compute?

Quick intuitive argument: we can compute the $L \to \infty$ limit (a.k.a. $\mathbb{R}^2$) the ground states are in correspondence with the massive vacua of the theory, which are given by the Bethe equations

$$e^{\frac{\partial \widetilde{W}_{eff}(\sigma_1, \cdots, \sigma_k, w)}{\partial \sigma_i}} = 1 \ , \forall i \in \{1, \cdots, k\} \ .$$

The action of the operator ∇_{t_0} becomes diagonal and its holonomies at a given vacuum can be computed to be (*c.f.* previous asymptotics)

$$p = e^{\frac{-2i\partial \widetilde{W}_{eff}(\sigma_1, \cdots, \sigma_k, w)}{\partial w}} \Big|_{\sigma_i = \text{vacua}}$$

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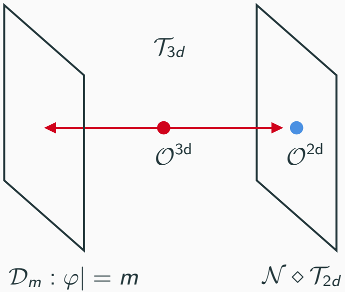
Thus we are left with simultaneous solutions to the equations

$$\left\{ e^{\frac{\partial \widetilde{W}_{eff}(\sigma_1, \cdots, \sigma_k, w)}{\partial \sigma_i}} = 1 \ , \quad p = e^{-2i \frac{\partial \widetilde{W}_{eff}(\sigma_1, \cdots, \sigma_k, w)}{\partial w}} \right\} \ , \ i \in \{1, \cdots, k\} \ .$$

Spectral Data I, $\lambda = 0$

In terms of 3d Coulob branches:

$$\left\{ e^{\frac{\partial \widetilde{W}_{eff}}{\partial \sigma_i}} = 1 \ , \quad v_-| = e^{-2i \frac{\partial \widetilde{W}_{eff}}{\partial \phi}} \right\} \ , \ i \in \{1, \cdots, k\} \ .$$



The spectral variety is the image of $\mathcal{N} \diamond \mathcal{T}_{2d}$!

Example

In the SQED[2] example, $X = \mathbb{P}^1$ and we have for $m = -2iw$

$$1 = e^{\frac{\partial \widetilde{W}_{eff}}{\partial \sigma}} = e^{-2\pi i \tau'}(\sigma + m)(\sigma - m)$$

together with

$$p = e^{\frac{\partial \widetilde{W}_{eff}}{\partial m}} \quad \Rightarrow \quad \sigma = \frac{m(p+1)}{p-1} \ ,$$

This gives

$$p^2 - 2(1 + 2e^{-2\pi i \tau'} m^2)p + 1 = 0$$

or

$$\left(p - \frac{\sqrt{m^2 + e^{2\pi i \tau'}} + m}{\sqrt{m^2 + e^{2\pi i \tau'}} - m}\right)\left(p - \frac{\sqrt{m^2 + e^{2\pi i \tau'}} - m}{\sqrt{m^2 + e^{2\pi i \tau'}} + m}\right) = 0 \ .$$

This is the spectral curve for a smooth $SU(2)$ monopole [Cherkis-Kapustin].

Spectral Data I, $\lambda \neq 0$

Recall that we are considering a $\mathbb{P}^1$ worth of supercharges Q_λ, and that these induce mini-holomorphic structures on the space of supersymmetric ground states. Consider

$$V := H^0(\mathbb{C}_{\beta_0}, E^0 := E|_{t_0=0})$$

Let F be parallel transport along ∇_{t_0}. Then

$$\mathcal{F} := \Phi^* \circ F : \underbrace{H^0(\mathbb{C}_{\beta_0}, E^0)}_{=:V} \mapsto \underbrace{H^0(\mathbb{C}_{\beta_0}, \Phi^* E^L)}_{\cong V}$$

where Φ^* is the pull-back by a $\beta_0 \mapsto \beta_0 - 2i\lambda L$. Thus, V is a $2\lambda iL$-difference $\mathbb{C}(\beta_0)$-module

$$\mathcal{F}(g(\beta_0) \cdot s) = \Phi^*(g(\beta_0)) \cdot \mathcal{F}s$$

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Thus, we can extract difference modules from Berry connections.

Spectral Data I, $\lambda \neq 0$

In fact, Mochizuki proves something along the following lines

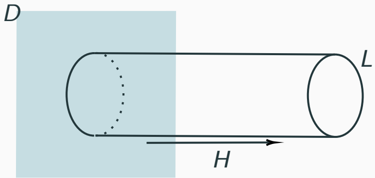
Mochikuzi's correspondence [Mochizuki]

There is a 1:1 correspondence between suitable $2i\lambda L$-difference $\mathbb{C}(\beta_0)$-modules and Cherkis-Kapustin monopoles

- This is quite deep and subtle. It is the culmination of decades of work on Hitchin-Kobayashi correspondences for monopoles
- The modules are endowed with structure that keeps track of the location of the singularities as well as the asymptotic behaviour at infinity and stability conditions
- Conditions at infinity require some adjustments to the mini-complex charts

Spectral Data I, $\lambda \neq 0$

Physically, we can consider the following configuration



$Q_\lambda, Q_\lambda^\dagger$ preserved

This will formally produce a state $|D\rangle$ in the QFT that is not necessarily a normalisable ground state (and therefore not an element of E). However by tt^* it generates a flat section of the Lax connection [Hori-Iqbal-Vafa]

$$\nabla_{t_0} |D\rangle = \nabla_{\bar{\beta}_0} |D\rangle = 0 \ .$$

Thus, we expect

$$\mathcal{F} |D\rangle = |D\rangle \ .$$

Spectral Data I, $\lambda \neq 0$

In [Ferrari-Zhang] we claim that this implies difference equations for so-called *brane amplitudes*. In fact, we can expand $|D(t_0 = 0)\rangle$ in a (λ-dependent) basis for E^0, say $|a^\lambda\rangle$

$$|D(t_0 = 0)\rangle \Rightarrow \sum_{a^\lambda} \langle a^\lambda | D \rangle \, |a^\lambda\rangle \ .$$

where $\langle a^\lambda | D \rangle := \langle a^\lambda | D(t_0 = 0) \rangle$.

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From $\mathcal{F} \, |D\rangle = |D\rangle$ we can derive an equation of the form

Difference equation for brane amplitude

For any basis $|a^\lambda\rangle$ there is an equation

$$\hat{p} \, \langle a^\lambda | D \rangle = G(\beta_0)_a^b \, \langle b^\lambda | D \rangle$$

where $\hat{p} := \Phi^*$ is a shift operator by $2i\lambda L$, and $G(\beta_0)_a^b$ is some matrix with entries in $\mathbb{C}(\beta_0)$.

Spectral Data I, $\lambda \neq 0$

Moreover, we can show that this matrix difference equation *quantises* the spectral curve of the monopole, which is

$$\mathcal{L} := \{(p, w) \in \mathbb{C}^* \times \mathbb{C} \mid \det(p - F(w)) = 0\} \ .$$

In fact, the operators $(\hat{p}, \hat{\beta}_0 := \beta_0 \cdot)$ satisfying

$$\hat{p}\hat{\beta}_0 = \hat{\beta}_0\hat{p} + 2i\lambda L\hat{p}$$

are quantised versions of the operators (p, w), and we have

Curve quantisation

As $\lambda \to 0$, we have

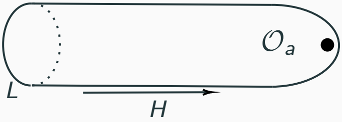
$$\lim_{\lambda \to 0} \mathcal{L}(G_a^b(\beta_0), \beta_0) = 0 \ ,$$

that is the eigenvalues of the matrix $G_a^b(\beta_0)$ satisfy the spectral curve equations.

Spectral Data I, $\lambda \neq 0$

Why?

We can take a basis $|a^\lambda\rangle$ that in the $\lambda \to 0$ limit reduces to the insertion of a chiral ring operator $\mathcal{O}_a$ at the tip of an A-twisted cigar



We can then consider *thimble* branes $|D^\alpha\rangle$, labelled by vacua α, which have known limiting behaviour. Combining these two things we obtain

$$\lim_{\lambda\to 0} \frac{(\Phi_1^*)^{-1} \langle a^\lambda | D^\alpha\rangle}{\langle \mathbf{1} | D^\alpha \rangle} = e^{-2i \frac{\partial W_{\text{eff}}^{(\alpha)}}{\partial w_i}} \mathcal{O}_a|_\alpha .$$

However, it is not straightforward to check these results directly (that is, to derive the difference equations in examples).

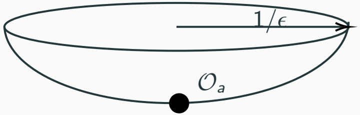
Spectral Data I, $\lambda \neq 0$

There is however another limit we can take, the so-called *conformal limit*

$$\lim_c : \quad \lambda \to 0, \quad L \to 0, \quad \frac{\lambda}{L} = \epsilon.$$

It is expected that in this limit, brane amplitudes degenerate into *hemisphere partition functions*

$$\lim_c \langle a^\lambda | D \rangle = \mathcal{Z}_D[\mathcal{O}_a, m - \epsilon x].$$



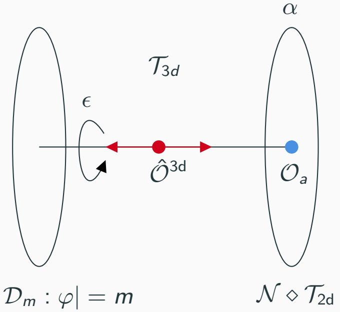
These are *exactly calculable* via localisation [Hori et al., ...]. The limit preserves the form of the difference equations

$$\hat{p}\, \mathcal{Z}_D[\mathcal{O}_a, m] = \mathcal{Z}_D[\mathcal{O}_a, m + \epsilon] = \widetilde{G}_a^b(m, \epsilon) \mathcal{Z}_D[\mathcal{O}_b, m] \ ,$$

as well as the fact that in the $\epsilon \to 0$ we should recover the classical curve. This appears to be new, and one can prove that it works! [Bonelli-Sciarappa-Tanzini-Vasko?]

Spectral Data I, $\lambda \neq 0$

In terms of the sandwich construction:



Left:

$$\langle \mathcal{D}_m | \, \hat{v}_- \, | \mathcal{N} \diamond \mathcal{T}_{2d}, \alpha, \mathcal{O}_a \rangle = \hat{p} \, \mathcal{Z}_\alpha[\mathcal{O}_a, m].$$

Right:

$$\langle \mathcal{D}_m | \, \hat{v}_- \, | \mathcal{N} \diamond \mathcal{T}_{2d}, \alpha, \mathcal{O}_a \rangle = \sum_b \widetilde{G}(m, \epsilon)_{ab} \, \mathcal{Z}_\alpha[\mathcal{O}_b, m].$$

Spectral Data I, $\lambda \neq 0$

Example

Let us consider the SQED[2] example

$$\mathcal{Z}_{D_1}[\mathbf{1}] = \oint_{\mathcal{C}_1} \frac{d\sigma}{2\pi i \epsilon} e^{-\frac{2\pi i \sigma \tau}{\ }} \, \Gamma\left[\frac{\sigma + m}{\epsilon}\right] \Gamma\left[\frac{\sigma - m}{\epsilon}\right],$$

$$\mathcal{Z}_{D_1}[\sigma] = \oint_{\mathcal{C}_1} \frac{d\sigma}{2\pi i \epsilon} e^{-\frac{2\pi i \sigma \tau}{\epsilon}} \, \Gamma\left[\frac{\sigma + m}{\epsilon}\right] \Gamma\left[\frac{\sigma - m}{\epsilon}\right] \sigma.$$

By means of an explicit computation we can for instance obtain

$$\left[\hat{p}^2 + 2\left(\frac{1+\frac{\epsilon}{2m}}{1+\frac{\epsilon}{2m}} + (2m+3\epsilon)(m+\epsilon)e^{-2\pi i \tau'}\right)\hat{p} + \frac{1+\frac{3\epsilon}{2m}}{1+\frac{\epsilon}{2m}}\right] \mathcal{Z}_{D_1}[\mathbf{1}] = 0 \ ,$$

which clearly is a quantum deformation of

$$p^2 - 2(1 + 2e^{-2\pi i \tau'} m^2)p + 1 = 0 \ .$$

Spectral Data II

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Consider for simplicity $\lambda = 1$, such that

$$M^{\lambda} = \mathbb{C}^* \times \mathbb{R} \ .$$

Q_λ is then a supercharge that squares to rotations of S^1_L, and its cohomology is expected to be related to the equivariant complex K-theory $K_T(X)$. Under our assumption, this takes the schematic form (Kirwan surjectivity) [Kirwan]

$$\mathrm{Spec}(K_T(X)) = \left(\bigsqcup_{\alpha \in X^T} \mathbb{C}^* \right) / \Delta \ .$$

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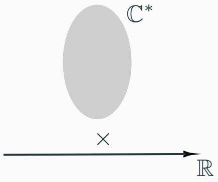
<u>Remark</u>: Q_λ is a dimensional reduction of the supercharge used to extract the elliptic cohomology variety of X from the Berry connection of a 3d $\mathcal{N} = 2$ QFT on $\mathbb{R} \times \mathcal{E}_\tau$ with Higgs branch X [Bullimore-Zhang,Dedushenko-Nekrasov].

Spectral Data II

Consider for simplicity SQED[2], so that E has of rank 2, and denote coordinates on $\mathbb{C}^* \times \mathbb{R}$ by $(z, \bar{z}, m_{\mathbb{R}})$.



We get holomorphic bundles $E^{m_{\mathbb{R}}}$ on $\mathbb{C}^*$, and the Bogomolny equations imply again that we can parallel transport holomorphic sections along $\mathrm{m}_{\mathbb{R}}$

$$[\nabla_{m_{\mathbb{R}}}, \nabla_{\bar{z}}] = 0 \ .$$

Spectral Data II

At $m_{\mathbb{R}} \to \infty$ there is a filtration $0 \subset \mathcal{L}^+ \subset E$ determined by sections that decay exponentially fast.

Similarly, at $m_{\mathbb{R}} \to -\infty$ there is a filtration $0 \subset \mathcal{L}^- \subset E$ determined by sections that decay exponentially fast. We can write

$$\psi^-(m_{\mathbb{R}}, z, \bar{z}) = f(z)\psi(m_{\mathbb{R}}, z, \bar{z}) + g(z)\psi^+(m_{\mathbb{R}}, z, \bar{z})$$

The locus $f(z) = 0$ supports “boundstates” –the support over which certain sections decay exponentially fast in both directions.

Remark: In a classic paper Hitchin [Hitchin] used a similar method to encode $SU(2)$ monopole solutions on $\mathbb{R}^3$ in terms of a *spectral curve* on its twistor space of lines TS^2

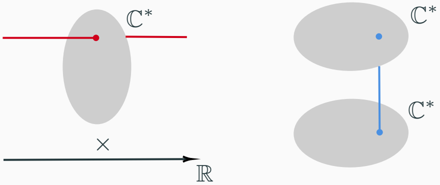
Spectral Data II

$K_T(\mathbb{P}^1)$ from Berry connections [Ferrari-Zhang]

We have that

$$\mathrm{Spec}(K_T(\mathbb{P}^1)) = (\mathbb{C}^* \sqcup \mathbb{C}^*)/(x_1 = x_2 = 1)$$

and so we can reconstruct $K_T(\mathbb{P}^1)$ from the spectral data of the Berry connection. The same holds for GKM varieties X.



Remark It is expected [Kontsevich-Soibelman] that this kind of spectral data (albeit with more intricate structures, such as filtrations at $0, \infty \subset \mathbb{C}^*$) is also sufficient to reconstruct the monopole solution.

Conclusion and future directions

Conclusions and future directions

- One can assign generalised periodic monopoles to to Kähler manifolds X (Berry connections for 2d $(2, 2)$ sigma-models with target X); the map by itself is interesting and poorly understood
- Different kinds of spectral data for these analytic objects are related to
    1. The action of a pure Coulomb branch algebra on $QH_T(X)$ that can be related to classic work [Teleman]
    2. The complex K-theory variety of $K_T(X)$
- In the conformal limit we obtain verifiable difference equations for vortex partition functions
- There should be a Riemann-Hilbert correspondence between the two kinds of spectral data [Kontsevich-Soibelman], and the connection to cohomology remains to be investigated
- All of these statements can be lifted to 3d, where spectral data for doubly periodic monopoles arising from physics should be realted to $QK_T(X)$ and elliptic cohomology

Thank you for your attention!